Photonic crystal with triangular stack profile

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Abstract

The enhanced localized reflectivity at a plane where the refractive index derivatives are discontinuous is increased using a periodic triangular stack. This optical system is studied using three different methods: a slowly varying refractive index approximate solution, numerical solutions to the nonlinear amplitude equation and the exact analytical solution using a matrix formalism. The field intensity within a photonic crystal with a periodic triangular profile is evaluated as a function of penetration depth and wavelength. These results are compared with the standard binary layered structures.

1. Introduction

An enhanced reflectivity and specific phase change upon reflection have been predicted for dielectric planes where the refractive index derivatives are discontinuous [1]. This reflectivity enhancement calculation was performed at normal incidence and takes place even if the refractive index itself is continuous. The reflectivity, at planes where the first order derivative is discontinuous, may reach values close to 1% if a convenient refractive index profile is chosen. The phase change upon reflection is \( \pm \pi/2 \), the sign depends on whether the refractive index slope increases or decreases [2]. In contrast, zeroth order derivative discontinuity corresponds to the Fresnel reflection of a dielectric step function, where reflectivities are around 4% (air–glass interface) and phase changes upon reflection are 0 or \( \pi \).

A periodic dielectric structure with triangular profile of the refractive index has been studied analytically using a semiclassical coupled wave theory [3]. An important advantage of wave solutions is, as the authors rightly assert, their superior physical insight. Recently, a sawtooth refractive index profile that encompasses the triangular profile has been exactly solved in terms of Bessel functions using the transfer matrix method [4]. However, the transfer matrix method does not lend itself to map the field or the intensity throughout the material. The field distribution is better represented in terms of Floquet–Bloch waves, since this procedure permits the evaluation of the field as it propagates through the periodic medium [5]. The light field distribution in photonic crystal structures has also been tackled using Fermat’s principle [6,7]. 1D photonic crystals have important technological applications in fast optical switching [8]. Graded multi-layer structures open up the possibility of shaping the refractive index’s profile, and hence tailor the reflection properties to specific designs.

The localized enhanced reflectivity due to the interface between two linear refractive indices with different slopes can be increased by a sequence of such derivative discontinuities. The triangular stack is the simplest scheme of this type. This triangular structure is a good experimental candidate to observe the reflectivity enhancement and phase shift upon reflection due to discontinuous first order derivatives.

In this communication, we explore the optical characteristics of the triangular stack using different theoretical methods. In Section 2, the reflection coefficient of a derivative discontinuity plane is derived in the amplitude and phase representation of waves. The nonlinear amplitude equation can be solved in power series of the slowly varying refractive index (SVRI) derivative, often called in the physics community, the JWKB approximation. At the discontinuity planes, where this SVRI scheme is not valid, the solutions at either side of the singularity are matched via field continuity conditions. In Section 3, the triangular stack is solved in the
SVRI approximation. First, the complex reflection coefficients for the two types of vertices needed to build up a triangular stack are obtained. Section 3.1 gives a rule of thumb to evaluate the reflectivity of a triangular quarter wavelength stack. Whereas in Section 3.2, the reflectivity as a function of incident wavenumber is evaluated. The numerical solution to the nonlinear amplitude equation, considering a triangular stack profile, is presented in Section 4. Once the amplitude and phase representation of the fields are understood in terms of an Ermakov pair, the implementation of these types of solutions is straightforward. Field intensity plots as a function of layer depth and wavelength can be readily produced in this scheme. In Section 5, the procedure outlined by Morozov et al. [4] is performed. The exact solution is compared with the amplitude numerical solution of the previous section. In Section 6, the triangle stack is compared with standard multilayer devices consisting of two alternating homogeneous media. Conclusions are drawn in the last section.

2. Reflection coefficient evaluated from the amplitude equation

Consider an isotropic, non-magnetic, transparent, dielectric medium with a linear response and no free charges, stratified in the z direction. Let the electric field \( E = E(z)e^{-ikt} \), where \( E \) represent monochromatic plane waves polarized in the x direction and propagation normal to the stratification planes. The non-autonomous ordinary differential equation (ODE) for the electric field is

\[
\frac{d^2E}{dz^2} + k^2 n^2(z)E = 0. 
\]  
(1)

The wave vector squared magnitude is \( k^2 = \omega^2 \mu_0 \epsilon_0 \) and the relative permittivity \( n(z) = \sqrt{\varepsilon(z)/\varepsilon_0} \). The amplitude \( A(z) \) and phase \( \phi(z) \) representation of the complex field is \( E(z) = A(z)e^{i\phi} \), where the amplitude and phase are real functions. Inserting this ansatz in the field equation leads to the nonlinear ordinary differential equation for the electric field amplitude [9]:

\[
\frac{d^2A}{dz^2} - \frac{Q^2}{A^3} = -k^2 n^2 A. 
\]  
(2)

The electric field solution \( E(z) \) is the total field at any propagation position \( z \). Whether there are counter-propagating waves or not and in which ratio is not discernible at this stage. Correspondingly, \( A(z) \) is the amplitude of the total field. The linearity of the field ODE guarantees superposition of field solutions. For the amplitude, it is necessary to invoke a nonlinear superposition principle in order to compose two or more amplitude contributions. In one dimensional problems, the amplitude and phase functions are related by the invariant:

\[
Q = A^2 \frac{d\phi}{dz}. 
\]  
(3)

This invariant can be derived from the scalar wave equation when a conservation equation is constructed from two linearly independent solutions [11]. The complementary fields associated with these two solutions exchange energy in a dynamical equilibrium both, in the time and spatial domains. In the one dimensional degenerate case, the conserved quantity becomes a constant.

Allow for an interval along the z-axis, where the medium is homogeneous, i.e. constant \( n \). Let the field solution be written as the sum of two waves with opposite phase:

\[
E(z) = A_+ \exp \left[ i(k_0 n z + \varphi_{0+}) \right] + A_- \exp \left[ -i(k_0 n z + \varphi_{0-}) \right]. 
\]  
(4)

where \( A_+, A_- \) are constant real amplitudes and \( \varphi_{0+}, \varphi_{0-} \) are real phase constants. Consider the semi-space where the wave is incident. In this region, these counter-propagating waves can be associated with the incident and reflected waves \( E(z) = E_{\text{Incident}} + E_{\text{Reflected}} \). Evaluation of the field derivative (4) gives:

\[
\frac{dE(z)}{dz} = i k_0 n E_{\text{Incident}} - i k_0 n E_{\text{Reflected}}. 
\]  
(5)

From these last two expressions, the reflected to incident fields ratio \( r(z) \) can be expressed in terms of the total field \( E \) and its first derivative [12]:

\[
r(z) = \frac{E_{\text{Reflected}}}{E_{\text{Incident}}} = \frac{i E k_0 n - \frac{dE}{dz}}{i E k_0 n + \frac{dE}{dz}}. 
\]  
(6)

where \( i \) stands for the imaginary unit. Alternatively, this ratio can be written in terms of the squared amplitude \( A^2 \) and its derivative

\[
\frac{n A k k_0 - Q - i A \frac{dA}{dz}}{n A k k_0 + Q - i A \frac{dA}{dz}}. 
\]  
(7)

where the invariant relationship (3) has been invoked. The ratio \( r(z) \) is interpreted as the complex reflection coefficient at a given plane of the incident semi-space. The square modulus of this quantity is the reflectivity \( R = r^2 \), it is constant in the incident semi-space before reaching the reflecting surface.

The nonlinear amplitude equation (2) can be approximately solved as a power series of the inverse wavenumber \( k_0^{-1} \) for a slowly varying refractive index (SVRI) but otherwise arbitrary function \( n(z) \) [13]:

\[
A_{\text{SVRI}} = \sum_{m=0}^{\infty} \frac{1}{k_0^m} A_m, 
\]  
(8)

where only the even order terms are non-vanishing. The \( A_m \) amplitude term retaining only the highest order derivative of the refractive index is [2]

\[
A_m = \left( \frac{(-1)^{m+1}}{m^{3/2}2^{m+1}} \right) \frac{d^m n}{dz^m} + O(e^m), 
\]  
(9)

for even \( m \). Solutions with higher precision can be analytically obtained by the evaluation of higher order terms in the slowly varying refractive index series expansion. Care should be taken with the convergence of the series. The field can then be evaluated from the amplitude solution together with the phase via the invariant relationship (3). In order to obtain the reflectivity at a discontinuity, the amplitude is evaluated just before and after the singularity where \( n(z) \) is analytic and slowly varying, Continuity conditions are imposed on these two amplitude solutions. The complex reflection coefficient for an isolated non-vanishing discontinuity in the mth order derivative of the refractive index at \( z_0 \) is given to first order by [2]

\[
R_{z_0} = \left( \frac{i m}{2m+3} \right) \frac{d^m n}{dz^m} \bigg|_{z_0} = \frac{d^m n}{dz^m} \bigg|_{z_{0-}} - \frac{d^m n}{dz^m} \bigg|_{z_{0+}}. 
\]  
(10)

where the light is incident from a medium where the refractive index mth derivative just before the discontinuity is \( d^m n/dz^m \bigg|_{z_{0-}} \) and just after the discontinuity is \( d^m n/dz^m \bigg|_{z_{0+}} \). The phase change upon reflection can be obtained from (10) recalling that \( im = e^{im \pi/2} \). The phase change can be summarized in the following
proposition:

For an interface with non-vanishing discontinuity in the mth order derivative of the refractive index, phase change upon reflection at the discontinuity plane is \(( (m+2) + s ) \pi \), where \( s = 1 \) for an increasing lowest order discontinuous derivative \( d^m n/\partial z^m|_{zz=d} > d^m n/\partial z^m|_{zz=-d} \) and \( s = 0 \) for a decreasing slope.

3. SVRI approximate analytical solution to the triangular stack

For an isolated discontinuity in a stack with triangular refractive index function as shown in Fig. 1, the first order derivative is discontinuous, i.e. \( m = 1 \) in Eq. (10). The reflection coefficient \( r_{iso} \) of a single isolated first order derivative discontinuity to leading order is [2]

\[
r_{iso} = \frac{i [n_{20} - \delta - n_{20} + \delta]}{4k_0 n^2},
\]

(11)

where \( n \) is the refractive index at the derivative discontinuity, \( n_{20} - \delta \) and \( n_{20} + \delta \) are the refractive index derivatives at both sides of the discontinuity plane. The reflection coefficient at the first triangular stack derivative discontinuity with higher refractive index \( n_2 \), labelled \( r_{iso1} \) in Fig. 1, is

\[
r_{iso1} = \frac{i [n_{11} - \delta] + [n_{21} + \delta]}{4k_0 n_2^2}.
\]

(12)

Other discontinuities lying in the upper part of the refractive index profile exhibit this same reflection coefficient, they are all labelled with odd numbers. The incident wave experiences a phase change upon reflection of \( \pi/2 \) at these discontinuities. The reflection coefficient of an isolated discontinuity at the lower part of the refractive index profile is

\[
r_{iso2} = \frac{i [n_{22} - \delta] + [n_{22} + \delta]}{4k_0 n_2^2}.
\]

(13)

The wave experiences a phase change upon reflection of \( -\pi/2 \) at even valued discontinuities. The phase difference between reflection coefficients of contiguous planes is \( \pi \). Notice that \( r_{iso1} \) and \( r_{iso2} \) have different magnitudes, and thus different reflectivity, due to the squared refractive index term in the denominator. Furthermore, the reflection coefficient is wavelength dependent and is inversely proportional to the wave number. These two issues should be contrasted with the square function where the reflectivity is identical for step-up or step-down and the phase change upon reflection is \( \pi \) or zero respectively. Nonetheless in either case, the phase difference between reflection coefficient of contiguous planes is \( \pi \), either for the triangular stack or the square function.

The refractive index derivative for both slopes of one triangle in the profile \( n(z) \), shown in Fig. 1, are

\[
n_i(n_2, n_1, d_1), \quad n_\pi(n_2, n_1, d_2)
\]

(14)

where \( n_i \) is the derivative with the positive slope side of the triangle and \( n_\pi \) at the negative slope, \( n_1 \) and \( n_2 \) are the lower and higher refractive indices of one layer, \( d_1 \) and \( d_2 \) are the layers thicknesses. The spatially dependent refractive index for both sides of the triangle is then

\[
n(z) = \int n_i dz = n_i z + b_i \quad \text{for the increasing side},
\]

(15a)

\[
n(z) = \int n_\pi dz = n_\pi z + b_\pi \quad \text{for the decreasing side},
\]

(15b)

where \( b_i \) and \( b_\pi \) are the ordinates for \( z = 0 \) and their values are different for every line segment of the profile. In particular

\[
b_j = n_1 - n_0 d, \quad b_\pi = n_\pi - (j + 1) n_0 d \quad \text{for } j = 0, 1, 2, \ldots \text{ and } d = d_1 + d_2
\]

is the length of the triangles’ base. The optical path within each layer is

\[
\Lambda_+ = \int_0^{d_1} n(z) dz = d_1 n_0 d + n_1 d_2, \quad \Lambda_\pi = \int_0^{d_2} n(z) dz = d_2 n_0 d + n_\pi d_1
\]

(16)

where \( \Lambda_+ \) and \( \Lambda_\pi \) refer to the increasing and decreasing refractive index layers respectively. For this linear refractive index function, the optical path is simply the average refractive index times the layer thickness.

The reflectivity of the stack has two contributions, one coming from the linear slope refractive index region (bulk) and another one coming from the enhanced reflectivity of the derivative discontinuities. Of these two, interference fringes arise mainly from the latter because the reflection from the bulk is spread out within the layer rather than sharply localized. The reflectivity contribution from the linear slope refractive index region diminishes as the slope becomes smaller [14]. The triangular stack shown in Fig. 1 is modelled by the function:

\[
n_\Lambda(z) = \begin{cases} n_2 - n_1 & \text{for } jd < z < jd + d_1 \\ n_1 - n_2 & \text{for } jd + d_1 < z < (j + 1)d \end{cases}
\]

(17)

The reflection coefficients (12) and (13) are then

\[
r_{iso1} = \frac{i [n_2 - n_1]}{4k_0 n_2^2} \left( \frac{n_2 - n_1}{d_1} - \frac{n_1 - n_2}{d_2} \right)
\]

(18a)

\[
r_{iso2} = \frac{i [n_1 - n_2]}{4k_0 n_2^2} \left( \frac{n_1 - n_2}{d_2} - \frac{n_2 - n_1}{d_1} \right)
\]

(18b)

For isosceles triangles the layers have equal thickness \( d_1 = d_2 = d/2 \), thus

\[
r_{iso1} = \frac{i (n_2 - n_1)}{k_0 n_2^2 d}, \quad r_{iso2} = \frac{i (n_1 - n_2)}{k_0 n_2^2 d}.
\]

(19)

3.1. Reflectivity estimate for a quarter wavelength stack

Consider an isosceles triangular stack with layers’ optical path equal to one quarter of a selected wavelength \( \lambda_p \), i.e. \( \Lambda = \frac{\lambda_p}{4} \). The layers thickness is \( d = \lambda_p / (n_2 + n_1) \). The absolute value of the refractive index slope in any layer evaluated form (17) is

![Fig. 1. Stack with triangular refractive index function. The z-axis depicts the penetration distance within the stack. The refractive index function is plotted in the ordinate axis.](image-url)
For \( n_2 = 2, n_1 = 1.5 \) and \( \lambda = \lambda_p \), the reflection coefficient for the two types of vertices are

\[
\begin{align*}
\text{r}_{\text{iso}1} = & \frac{(n_2^2 - n_1^2)}{2 \text{en}_1^2} = 0.0696i, \\
\text{r}_{\text{iso}2} = & \frac{(n_1^2 - n_2^2)}{2 \text{en}_1^2} = -0.124i.
\end{align*}
\]

The percentage reflectivity at these vertices is \( R_{\text{iso}1} = \text{r}_{\text{iso}1} \times 100 = 0.48\% \) and \( R_{\text{iso}2} = \text{r}_{\text{iso}2} \times 100 = 1.53\% \). The reflection coefficients for the junction planes with the surrounding homogeneous medium are evaluated in a similar fashion

\[
\begin{align*}
\tilde{n}_1 = & \frac{(n_2^2 - n_1^2)}{4 \text{en}_1^2} = -0.0619i.
\end{align*}
\]

On the other hand, recall that for a periodic \( z/4 \) stack of homogeneous layers with alternating refractive indices \( n_1 \) and \( n_2 \) (discontinuous \( n(z) \)), the reflectivity can be obtained with the transfer matrix method. For an odd number of reflecting planes [15, Sect. 1.6, Eq. (96)]

\[
R(\lambda_p) = \text{tr}^* = \left( \frac{n_1 n_2}{n_1 n_2} \right)^N - n_1 n_2 \left( \frac{n_1 n_2}{n_1 n_2} \right)^{N-1}.
\]

where \( N \) is the number of periods, \( n_1 \) and \( n_2 \) are the high and low refractive indices respectively, \( n_1 \) and \( n_2 \) are the incident medium and substrate indices. Care has been taken to maintain whether a reflectivity arises from the step \( n_1 \) to \( n_2 \) or the other way around. This precaution is unnecessary for binary media because the reflectivity magnitude is the same regardless of whether it is a step up or a step down. However, this is not the case for a spike up or spike down of the triangle stack profile as we have already seen in the previous section. Although expression (23) is an exact result only for alternating homogeneous \( z/4 \) layers, it can be used as a first and simple approximation to estimate the maximum reflectivity for a stack with an odd number of derivative discontinuities. To this end, recall that the quotients \( n_1/n_2 \) and \( n_2/n_1 \) can be written in terms of the reflection coefficients of the individual interfaces. The reflection coefficient at normal incidence is \( x_0 = (n_0 - n_a)/(n_0 + n_a) \) for a wave travelling from \( n_0 \) to \( n_a \). From this expression, the quotient of refractive indices can be written as \( n_1/n_2 = (1 - n_0)/(1 + n_0) \). Making the substitutions \( n_1 \rightarrow n_1 \), \( n_2 \rightarrow -n_1 \), \( n_1 \rightarrow -n_2 \), and \( n_2 \rightarrow -n_1 \), the reflectivity for a triangular stack is

\[
R(\lambda_p) = \left( \frac{1 - [n_1]}{1 + [n_1]} \right)^N - \frac{1}{1 + [n_1]} \left( 1 + \left[ \frac{n_1 n_2}{n_1 n_2} \right] \right)^{N-1}.
\]

This expression can be written in the usual fashion where the leading terms in numerator and denominator are set to one

\[
R(\lambda_p) = \left( \frac{1 - [n_1]}{1 + [n_1]} \right)^N - \left( \frac{1}{1 + [n_1]} \right)^{N-1}.
\]

Even for binary system with a small \( r_{\text{iso}} \), the resulting reflectivity can be very large for numerous layers. If the number of derivative discontinuities is sufficiently large, high reflectivities can also be achieved. Therefore, an \( n(z) \) profile consisting of a triangle stack, as shown in Fig. 2, can be used to build photonic crystals as well as rugate mirrors or filters [16]. For the profile defined in (17), with \( N = 10 \), and \( n_1 = n_1 = 1.5 \) and \( n_2 = 2 \), an estimate of the reflectivity using (25) is \( R(\lambda_p) = 92\% \).

### 3.2. Reflectivity as a function of wavelength in SVRI approximation

In order to evaluate the optical system behaviour as a function of wavelength, not only \( \lambda = \lambda_p \), we can follow the matrix method procedure that leads to Eq. (25) but allowing for the inhomogeneous layers of the triangular stack. The electric field amplitude of a wave, propagating in one direction, within a gradually varying refractive index layer can be expanded in an even power series of \( k_0^{-1} \) [2]. The invariant \( Q \) in (3) is chosen with unit amplitude at a constant refractive index region as \( Q = k_0 \). To second order, the SVRI amplitude solution is

\[
A_{\text{SVRI}}(z) = n^{-1/2} + k_0^{-1} \left\{ \frac{3}{16} n^{-9/2} \frac{dn}{dz} + n^{-7/2} \frac{d^2n}{dz^2} \right\}.
\]

We can thus construct the electric field solution for counter-propagating waves within this layer

\[
E_{\text{SVRI}}(z) = A_{\hat{z}} e^{i \phi_{\text{SVRI}}} + A_{\hat{z}} e^{-i \phi_{\text{SVRI}}},
\]

where \( A_{\hat{z}} \) and \( A_{\hat{z}} \) are complex constants and the phase is given from the invariant relationship (3) as

\[
\phi_{\text{SVRI}} = k_0 \int_0^z \frac{dz}{A_{\text{SVRI}}^2}.
\]

It is possible to expand the integrand in a power series of \( k_0^{-1} \) but, because of the extra \( k_0 \) factor, the result for \( \phi \) is a series with odd powers of \( k_0^{-1} \). Retaining terms up to second order in \( k_0^{-1} \) means that the only non-vanishing term that remains for \( \phi \) is first order

\[
\phi_{\text{SVRI}} = k_0 \int_0^z n \ dz = k_0 A(z),
\]

where \( A(z) \) is the optical path. Imposing the boundary conditions in the matrix formalism [15, Sect. 1.6, Eq. (28)], the matrix \( \mathbf{m}_{\text{SVRI}} \) entries for a layer extending from \( z = 0 \) to \( z = d/2 \) are

![Fig. 2. Stack with isosceles triangular refractive index function plotted in the ordinate axis. The refractive index is a continuous function but its derivative is discontinuous. Contributions from the discontinuities, labelled \( r_{\text{iso1}} \) and \( r_{\text{iso2}} \), play a dominant role in the optical characteristics of the film.](image-url)
where the prime stands for the derivative with respect to \( z \). The values of \( A_{SVRI} \) and \( A_{SVRI} \) for an increasing linear refractive index layer, at \( z=0 \) and \( z = d/2 \) are

\[
A_{SVRI}(0) = n_1^{-1/2} - k_0^2 \left[ \frac{3}{16} n_1^{-3/2} \left( \frac{2}{\lambda p} \right) (n_2^2 - n_1^2) \right]^2,
\]

\[
A_{SVRI}(d/2) = n_2^{-1/2} - k_0^2 \left[ \frac{3}{16} n_2^{-3/2} \left( \frac{2}{\lambda p} \right) (n_2^2 - n_1^2) \right]^2,
\]

\[
A_{SVRI}(0) = -\frac{1}{2} n_1^{-3/2} \left( \frac{2}{\lambda p} \right) (n_2^2 - n_1^2) + k_0^2 \left[ \frac{27}{32} \right],
\]

\[
A_{SVRI}(d/2) = -\frac{1}{2} n_2^{-3/2} \left( \frac{2}{\lambda p} \right) (n_2^2 - n_1^2) + k_0^2 \left[ \frac{27}{32} \right].
\]

The substitution \( n_2 \rightarrow n_1 \) yields the values for a decreasing linear refractive index layer. The advantage of the transfer matrix formalism is that it is possible to build stratified media by multiplying single layer matrices. For a single triangle \( m_{SVRI}^{\text{triangle}} = m_{SVRI}^{\text{triangle}} m_{SVRI}^{\text{triangle}} \), where \( m_{SVRI} \) is the matrix for an increasing refractive index interval and \( m_{SVRI} \) the matrix for a decreasing one. For a triangle stack \( m_{SVRI}^{\text{stack}} = \left( m_{SVRI}^{\text{triangle}} \right)^N \), where \( N \) is the number of triangles in the stack. The \( N \)th matrix power can be evaluated from the unimodular matrix power expression given by Eq. (6) in [4]. This result can be viewed as a consequence of the Floquet theorem. This product can also be written in terms of the Chebyshev polynomials of the second kind \( U_k \), \( m = m^{\text{N}} = U_{N-1}(a) m - U_{N-2}(a) I \), where \( a = \frac{1}{2}(m_{11} + m_{22}) \) and \( I \) stands for the identity matrix. Following the transfer matrix method prescription to find the reflection coefficient [15, Sect. 1.6, Eq. (49)] yields

\[
m_{SVRI} = \frac{m_1(M_1^{SVRI} + n_1 M_2^{SVRI}) - (n_1 M_2^{SVRI} + M_1^{SVRI})}{m_1 (M_1^{SVRI} + n_1 M_2^{SVRI}) + (n_1 M_2^{SVRI} + M_1^{SVRI})},
\]

where \( M_{ij}^{SVRI} \) are the matrix entries for \( m_{SVRI}^{\text{stack}} \). Reflectivity follows immediately since \( R = r^r \).

### 4. Amplitude numerical solution

The amplitude equation (2) can be solved numerically for the triangle stack profile described by Eq. (17) and shown in Fig. 2. The solutions give the field amplitude as a function of position within the stack and outside it. To plot each solution, 3800 points were evaluated in a \( \Delta z = 4 \lambda p \) range which includes the triangle stack and half a wavelength of the homogeneous medium before and after this structure. In order to find the reflectivity it suffices to consider the amplitude solution and its derivative at the incident plane where the stack begins or an arbitrary plane before the stack \((z < 0)\). In our evaluations, the \( z = -0.5(\lambda p) \) plane was used. The reflectivity can then be computed from (7):

\[
R_{\text{numerical}}(k_0) = \left( \frac{k_0 A n - Q}{A} \right)^2 + A^2.
\]

The derivative of the amplitude at \( z_i \) is evaluated from the difference of two contiguous amplitudes of the mesh, \( A_i = (A_i - A_{i-1})(z_{i+1} - z_{i-1}) \). The amplitude equation can be solved for different values of \( k_0 \) or what is equivalent, different values of the triangles’ thickness \( d \). The value of \( k_0 \) (or alternatively \( z_0 \)) was increased by 0.01 rad/\( \lambda p \) (or 0.002\( \lambda p \)) with each iteration. The solutions were found via a finite difference method. It is remarkable that although the amplitude differential equation is nonlinear, its numerical behaviour is extremely docile. The numerical results for a wide range of frequencies are shown in Fig. 3 along with the analytical SVRI approximate solutions.

Constructive and destructive interference is responsible of the local maxima and minima. A general reduction in reflectivity is seen as the wavenumber increases, since the modulus of the reflection coefficient for isolated reflecting planes is inversely proportional to \( k_0 \). The SVRI approximation works better when the profile is soft, i.e. when changes of the refractive index occur within distances that are large compared to the wavelength. Although the profile is the same for all frequencies, for larger wavenumbers it is relatively more gradual, so the SVRI approximation is more adequate for large \( k_0 \). This situation is clearly seen in Fig. 3, for \( k_0 < 3 \) the SVRI approximation departs greatly from the numerical result.

The solutions to the amplitude equation (2) are obtained as a function of position and can be evaluated for different wavelengths. They can be renormalized to render a bi-dimensional intensity map, as the one shown in Fig. 4. An intensity map can be visualized as colour plot for \( A^2 \) as a function of depth \( z \) and wavelength \( \lambda \). Incident waves are set to have the same amplitude for all wavelengths. The map shows the energy distribution within the medium revealing hot spots that can be relevant in high intensity applications [16]. The abscissa is now wavelength rather than wavenumber (as shown in Fig. 3), so wavelength increases to the right. The wavelength range is just a small part of that plotted in Fig. 3. It is chosen to include \( \lambda = \lambda p \) that corresponds to \( 2 \pi \) in the wavenumber plot. Depending on the application both types of plots are widely used. The incident wave comes from the lower side of the figure. At the centre frequency the \( \lambda p/4 \) stack is an
5. Exact analytical solution

An exact solution to the quadratic permittivity (linear refractive index) has recently been presented by Morozov et al. [4]. Consider the electric field differential equation (1), with quadratic permittivity:

\[ \varepsilon(z) = n^2(z) = \left(n_1 + \left(\frac{n_2 - n_1}{2}\right)^2 \right), \]

(34)

corresponding to a linear and increasing \( n(z) \) profile. The solutions are \( E = \xi^{\nu/2} J_{\nu}(\xi) \), where \( J_{\nu/2} \) are Bessel functions and

\[ \xi(z) = -\frac{k_0 d}{4(n_2 - n_1)} \left[ n_1 + (n_2 - n_1) \frac{2\pi^2}{\lambda^2} \right]. \]

Morozov et al. [4] used this result to find the reflectivity and transmittance of a sawtooth profile and outlined that this same procedure is appropriate to work out a triangular stack profile. Following their suggestion, the \( m_{\alpha} \) entries for an increasing refractive index layer, extending from \( z=0 \) to \( z=d/2 \), are

\[ m_{\alpha11} = \frac{\sqrt{2\xi d n_2}}{2n_1} \left( J_{1/4}(\xi_0) J_{-3/4}(\xi_f) + J_{1/4}(\xi_0) J_{3/4}(\xi_f) \right) \]

\[ m_{\alpha12} = -i \frac{\xi d n_2}{2n_1} \left( J_{1/4}(\xi_0) J_{1/4}(\xi_f) - J_{1/4}(\xi_0) J_{-1/4}(\xi_f) \right) \]

\[ m_{\alpha21} = i \frac{\xi d n_2}{2} \left( J_{3/4}(\xi_0) J_{-1/4}(\xi_f) - J_{3/4}(\xi_0) J_{1/4}(\xi_f) \right) \]

\[ m_{\alpha22} = \frac{\sqrt{2\xi d n_1}}{2n_2} \left( J_{1/4}(\xi_0) J_{1/4}(\xi_f) + J_{-3/4}(\xi_0) J_{-1/4}(\xi_f) \right) \]

(35)

where \( \xi_0 = \xi(0) \) and \( \xi_f = \xi(d/2) \). To achieve compact expressions for the matrix elements, the following standard relations among Bessel functions [17], were used:

\[ \frac{d}{d\xi} \left( \xi J_{\nu}(\xi) \right) = \xi J_{\nu-1}(\xi), \quad \frac{d}{d\xi} \left( \xi^{-1} J_{\nu}(\xi) \right) = -\xi^{-1} J_{\nu+1}(\xi). \]

The matrix \( m_{\alpha} \) entries for a decreasing refractive index \( n(z) \) interval are obtained from (35) if the \( n_2 \leftrightarrow n_1 \) exchange is made. The reflectivity of a periodic triangular stack can then be obtained using the \( W \) transfer matrix method as described by Morozov et al. [4]. Following their procedure, the product of a matrix \( m_{\alpha} \) for a positive slope times a matrix \( m_{\beta} \) for the negative slope produces one layer. The transfer matrix for a stack with \( N \) layers is then

\[ M_N = (m_{\alpha} m_{\beta})^N \]

or

\[ M_N = U_{N-1}(a)(m_{\alpha} m_{\beta}) - U_{N-2}(a)1, \]

(36)

where \( U_N(z) \) are the Chebyshev polynomials of the second kind and \( a = \frac{1}{2}(m_{11} + m_{21}) \). The reflection coefficient is then given by

\[ R_W = \frac{m_{11}[M_{11} + M_{12} - (n_1 M_{22} + M_{21})]}{n_{11}[M_{11} + n_{12} M_{22} + (n_1 M_{22} + M_{21})]}, \]

(37)

where \( M_{ij} \) are the elements of the matrix \( M_N \) given by (36).

The exact analytical solution produces a reflectivity spectrum \( R_W(k_0) \) that, when plotted at the scale of Fig. 3, shows no difference from the numerical result. Although reflectivity differences

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**Fig. 3.** Percentage reflectivity as a function of wavenumber normalized to a principal wavelength \( \lambda_p \), numerical evaluation (black line) and SVRI approximate result (red dotted line). (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this paper.)

**Fig. 4.** Field intensity plot for a triangular stack mirror, with \( 8 \sim 90 \% \) at the central wavelength. The ordinate represents the penetration depth while the abscissa represents the incident light wavelength (light is incident from below in the figure). Both quantities are normalized to the so-called principal wavelength \( \lambda_p \). The triangular stack refractive index profile, depicted in Fig. 2, is superimposed on the right side for reference. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this paper.)

Efficient mirror. The lower part of the intensity map at \( \lambda_0/\lambda_p = 1 \) reveals a large amplitude spatial modulation due to the counter-propagating waves with similar amplitude. The transmitted intensity at \( \lambda_0/\lambda_p = 1 \) is very small beyond \( z/\lambda_p = 2 \), i.e. the transmitted wave becomes negligible. The high reflectivity plateau as a function of wavelength is visualized by the width of the low intensity transmitted wave (purple region in the middle upper part of Fig. 4). At \( \lambda_0/\lambda_p \approx 0.6 \), the modulation is rather soft, revealing a very low reflection coefficient at this wavelength. The optical path is evaluated by multiplying \( z/\lambda_p \) by the average refractive index. In Figs. 2 and 4, the average refractive index is 1.75. Ten triangles involve two quarter wavelength layers, thus the overall optical path is 5 wavelength units. The stack depth is then 5/1.75 \approx 2.86 in \( \lambda_p \) units.
are small, they can be highlighted by plotting $R_{W\text{ matrix}} - R_{\text{numerical}}$ as shown in Fig. 5. The plot reveals a slight left-shift of $R_{\text{numerical}}(k_0)$ relative to $R_{W\text{ matrix}}(k_0)$, that grows proportional to $k_0$. This $\Delta k_0$ shift never exceeds a reflectivity of $1 \times 10^{-2}$ in the region of interest. This error can be made arbitrarily small by decreasing the sampling interval. The close agreement of the numerical and exact analytical results exhibit the good quality of the numerical method.

6. Comparison with a DBR refractive index profile – binary approximations

A typical distributed discrete Bragg reflector (DBR) has a step refractive index profile as shown in Fig. 6a. Since discontinuities in the derivative of $n(z)$ also behave like reflecting planes, it is possible to construct devices similar to DBR mirrors by placing them periodically. A profile of this kind is shown in Fig. 6b. To compare the behaviour of a DBR and a “triangle stack function”, planes with the same periodicity and similar reflectivity are chosen to build both profile types. For the DBR, $n_{\min} = 1.5$ and $n_{\max} = 1.821$ are chosen, so that the reflection coefficient for one interface is $r = ((n_{\max} - n_{\min})/(n_{\max} + n_{\min})) = 0.0967$. This value is close to the average magnitude between $n_{a1}$ and $n_{a2}$ in (21). Numerical solutions to the amplitude equation (2) for the triangular stack and the DBR profiles can be readily obtained for different wavelengths. Thereafter, the overall reflectivity is evaluated using Eq. (6) and plotted as a function of wavelength in Fig. 6c. The “triangle stack function” mirror achieves a similar reflectivity as a DBR for the main wavelength $\lambda_0/\lambda_p = 1$. The triangular stack side lobes are smaller for shorter wavelengths ($\lambda < 1$) and slightly higher for greater wavelengths ($\lambda > 1$). Eq. (11) shows that single vertex reflectivity is proportional to the square of the wavelength, this explains the asymmetric height of the side lobes in Fig. 6c. Other layered devices such as dielectric chirped mirrors could also be produced with this design by conveniently changing the triangles pitch.

Recall that the vertices of the triangle stack at $n=2$ and $n=1.5$ have different reflection coefficients due to the refractive indices in the denominators of Eqs. (12) and (13). In contrast, all DBR interfaces have the same reflectivity. Eq. (25) allows us to give a rough estimate for the triangle stack function reflectivity. For a stack of 10 triangles, the peak reflectivity estimate is $R \sim 92\%$. The numeric result at $\lambda_0 = \lambda_p$ is $R = 90\%$. The rough estimate is then slightly higher than the numerical calculation but certainly close to it. Experimental confirmation of the optical behaviour of these triangular stacks is most desirable.

A DBR intensity map, similar to that produced for the triangular stack, is shown in Fig. 7. The map shows a colour plot of $A^2$ as a function of depth $z$ and wavelength $\lambda$ normalized to $\lambda_p$ at a quarter wavelength. The intensity plot exhibits a high reflectivity plateau at the designed wavelength, similar to the one obtained for the triangular stack. However, at $\lambda_0/\lambda_p = 0.6$ the modulation is much larger than that of the triangular stack. So, with the appropriate design parameters, the triangular stack could provide higher transmittance at a certain wavelength and very high reflectivity at another wavelength. This feature can be exploited in optical components where high reflectivity is required at one wavelength and full transmission is required at another wavelength. Longitudinally pumped lasers such as diode pumped Ti:Sa systems certainly come to mind where the pump wavelength should go through but the lasing wavelength should be reflected.

7. Conclusions

A quarter wavelength periodic triangular structure increases the reflectivity due to the refractive index first order discontinuities of an otherwise continuous refractive index function.
Such structures with linear refractive index variation are well within the technological expertise of laboratories growing rugate thin films [18]. These structures should exhibit a $\pm \pi/2$ phase change upon reflection, the sign depending on whether there is an increasing or decreasing slope in the vicinity of the derivative discontinuity. This reflectivity enhancement as well as the phase change upon reflection could be experimentally tested with a triangular stack profile.

A rough estimate of the maximum reflectivity for a quarter wavelength stack has been presented using Eq. (25), where the isolated reflectivities are given by (21). An approximate analytical solution to a periodic triangular structure has been obtained in the slowly varying refractive index (SVRI) approximation. This estimate can be used to evaluate the elements of the transfer matrix, and in turn, to evaluate the reflectivity at different wavenumbers. In the particular case of the triangular function, it has recently been shown that an exact analytical solution is possible [4]. Alternatively, the nonlinear amplitude equation can be solved numerically, not only for the triangular profile but also for a wide variety of functions. The three methods and the rough estimate give consistent results, although the SVRI approximation obviously diverges when the refractive index variation is large compared to the wavelength. The plots of the exact analytical result and the numerical evaluation of the reflectivity as a function of wave-number are almost identical. Departures between them are not larger than 1% in the worst case for a wavenumber span of more than one order of magnitude.

The isolated reflectivity of a first order derivative discontinuity is about four or five times smaller than that of an abrupt change in the refractive index. Nonetheless, for a large number of layers, high reflectivities can also be obtained with the triangular structure build up by a succession of quarter wavelength layers. The side lobes of the triangular stack are asymmetric due to the wavelength dependence of the reflection coefficient, $r_{130}$. A single isolated first order derivative discontinuity. Recall that side lobes have also been inhibited by apodization with sinusoidal rugate profiles. A similar apodization technique can be implemented with the triangular profile [16]. A photonic crystal with triangular profile allows for asymmetric modifications of the reflectivity as a function of wavenumber. This feature extends the optical flexibility of such photonic structures.

On the other hand, the triangular stack exhibits wavelength regions where the reflectivity is exceptionally low compared with a conventional binary DBR. See the lack of modulation at approximately 0.6$n_0/\lambda_p$ in Fig. 4 and almost null reflectivity in Fig. 6c. These results suggest that this triangular structure could be of interest for anti-reflection coatings. The stack shown in the plots has the same initial and final refractive indices. It is thus a free-standing like film. The phase shift upon reflection depends on whether the last layer next to the substrate has higher or lower refractive index. Specific AR coatings will require the substrate as well as the initial refractive index step information to produce an appropriate design.

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**References**


