Phase change of light reflected by a discontinuity in the derivatives of the refractive index

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The amplitude and phase representation of classic plane electromagnetic waves is used to model light propagating through transparent stratified media, with a continuous refractive index profile. Numeric solutions for the nonlinear amplitude equation at normal incidence are obtained. Discontinuities of the refractive index derivatives exhibit reflection enhancement. Depending on the order of the discontinuity, the phase change upon reflection obtained from the numerical results can be ±π/2 in addition to the usual 0, or ±π.

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1. Introduction

The problem of inhomogeneous thin films has been approached by many authors. The potentialities of these coatings for optics have intrigued scientists for over a century and they are still a matter of research both experimentally and theoretically [1–11]. Discontinuities in the refractive index derivatives influence strongly the reflectivity of continuously varying refractive index films, this fact has been noticed since early works [4,6,7]. In these works the discontinuity planes were not treated individually as localized reflectivity sources. Two discontinuity planes were always inserted for each stratified medium, merely to model the two borders of a thin film.

By introducing only one derivative discontinuity, in an otherwise smooth and gradual refractive index profile, we may find its properties as a reflection plane. The amplitude and phase representation will be used to model light propagation through transparent stratified media [12,13]. Numeric solutions for the nonlinear amplitude equation at normal incidence will be obtained, from these solutions reflectivity and phase change upon reflection will be inferred. From the analysis of discontinuous derivatives at different orders as well as many different profile functions, a conjecture can be established.

1.1. Amplitude equation

Let us take a close look to this physical phenomenon, regarding reflection of light on a plane surface with discontinuous refractive index or discontinuous index derivatives. Important works trying to describe propagation of electromagnetic waves through stratified media have already been published. Somehow, these works use approximate expressions, most of them being for either a slowly or abruptly varying refractive index [3–5,7]. These approximations allow analytical expressions to be written for the fields and the reflectivity. Note that these early works were published before commercial computers were widely used by scientists and regular population, numerical analysis was not as easily performed as it is today. In the present work, no approximations of this kind are intended, just those inherent to the use of numerical methods.

For that purpose a differential equation for the electric field amplitude will be convenient [12–15]. It is a nonlinear ordinary differential equation of the Ermakov–Milne–Pinney type [16–18].

Consider Maxwell’s equations, in an isotropic, transparent, dielectric medium with a linear response and no free charges, let the electric permittivity and magnetic permeability vary spatially. The second order differential equation for the electric field is

\[ \nabla (E \cdot \nabla \ln \varepsilon) + \nabla \nabla \ln \mu \times (\nabla \times E) = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} - \nabla^2 E, \quad (1) \]

where \( \epsilon \) and \( \mu \) represent the electric permittivity and magnetic permeability, respectively. Let \( z \) be the direction of stratification, so that \( \epsilon \) and \( \mu \) will depend only on \( z \). Consider a monochromatic plane wave with frequency \( \omega \), and linear TE polarization, say, in the \( x \) direction \( E = E_0(y,z)e^{-i\omega t}i \). Eq. (1) leads to [19]

\[ \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \omega^2 \mu E_x = \frac{d(\ln \mu) E_x}{dz} \frac{\partial E_x}{\partial z}. \quad (2) \]

For normal incidence and non magnetic media \( \mu = \mu_0 \). This PDE becomes the non autonomous ODE

\[ \frac{\partial^2 E_x}{\partial z^2} + K_0^2 n^2 E_x = 0, \quad (3) \]
where \( k_0^2 = \alpha^2 \mu_0 \varepsilon_0 \) and the refractive index is \( n = \sqrt{\varepsilon / \varepsilon_0} \), \( \varepsilon_0 \) is the electric permittivity of vacuum. Now, consider a complex \( \varepsilon \), namely \( \varepsilon = \varepsilon_0 e^{\delta} \), where the amplitude \( A \) and phase \( q \) depend on \( z \). We restrict the problem to non-absorbing media, so we will assume the refractive index to be a real quantity. Substitution in Eq. (3) and separation of real and imaginary parts render:

\[
d^2 A \left( \frac{dq}{dz} \right)^2 = -k_0^2 n^2 A, \tag{4}
\]

\[
2 \left( \frac{dA}{dz} \right) \left( \frac{dq}{dz} \right) + A \frac{d^2 q}{dz^2} = 0. \tag{5}
\]

Eq. (5) can be readily integrated to obtain an invariant quantity given by

\[
Q = A^2 \frac{dq}{dz}. \tag{6}
\]

A nonlinear ordinary differential equation for the amplitude is obtained upon substitution of this result in Eq. (4)

\[
d^2 A \frac{Q^2}{A^2} - A = -k_0^2 n^2 A. \tag{7}
\]

This is an Ermakov–Milne–Pinney type equation. In order to work with a dimensionless amplitude function let us introduce \( A_d = A/\sqrt{k_0/Q} \), then Eq. (7) can be rewritten

\[
\frac{d^2 A_d}{dz^2} - \frac{1}{A_d} \left( \frac{dA_d}{dz} \right)^2 = -n^2 A_d. \tag{8}
\]

This is the ordinary differential equation for the electric field amplitude. It is not common to turn a linear differential equation into a nonlinear ODE, but Eq. (8) poses no challenge to be solved numerically if \( A_d \) is real and \( n \) is bounded. Also, initial conditions are easily imposed having a clear physical meaning and finally, interpretation of the solutions is straightforward. The finite difference method for numerically solving ordinary differential equations is used, with single precision floating point numbers.

1.2. Interpretation of the solutions

A constant \( n \) represents a homogeneous medium, in this case the solutions of Eq. (8) are known and must be of the form \([12–14,20]\):

\[
A_d (z) = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos (2k_d z + \delta)}. \tag{9}
\]

The field amplitude \( A_d \) is produced by the superposition of two counter propagating waves, with individual constant amplitudes \( A_1 \) and \( A_2 \). The constant \( \delta \) is the phase difference between both waves at \( z = 0 \). There is a restriction for these amplitudes if Eq. (8) is to be satisfied:

\[
(A_1^2 - A_2^2)^2 = \frac{1}{n^2}. \tag{10}
\]

This homogeneous medium solution \( A_d (z) \) oscillates periodically if both, \( A_1 \) and \( A_2 \), are nonzero. Maxima \( A_{\text{max}} \) and minima \( A_{\text{min}} \) occur when the incoming and outgoing waves are in or out of phase, respectively. These extrema can also be related to the ratio

\[
r = A_2 / A_1 \quad [12,13]:
\]

\[
r = \frac{A_{\text{max}} - A_{\text{min}}}{A_{\text{max}} + A_{\text{min}}}. \tag{11}
\]

If there is only one wave propagating, \( A_1 \) or \( A_2 \) is zero and \( A_d \) is constant, particularly

\[
A_d^2 = \frac{1}{n^2}. \tag{12}
\]

To model a single interface, \( n(z) \) must be a monotonic continuous function evolving from \( n_1 \) to \( n_2 \). Far from the interface \( n \) should be almost constant. To evaluate the reflectivity, the convenient initial condition is a single transmitted wave through the second medium, so the incident light is assumed to come only from the first medium side. This implies that the solution in the second medium, far from the interface, is almost constant \( A_d = 1/\sqrt{n_2} \). Under this condition, the amplitude oscillations in the first medium, far from the interface, will exhibit the medium’s reflectivity: \( R = r^2 \). Given the indices \( n_1 \) and \( n_2 \), we expect the reflectivity to depend on the interface abruptness. This property can be characterized by the distance \( D \) through which the index varies from \( n_1 + (1/20)Dn \) to \( n_1 - (1/20)Dn \), so that parameter \( D \) can be thought as the interface thickness, corresponding to 90% of the index change as shown in Fig. 1.

In an earlier paper [13] studies of the interface reflectivity, given different profiles with varying thicknesses, have been done. All profiles were continuous, but some were piecewise defined and their derivatives were discontinuous. For “hard” interfaces, meaning \( D \approx \lambda/2 \), reflectivity was close to the Fresnel result.

 regardless of the profile type. For “softer” interfaces, \( D \approx \lambda/2 \), the reflectivity of almost all \( n(z) \) profiles fell to less than 6% of the Fresnel result. The reflectivity for the analytic profiles dropped monotonically for \( D \approx \lambda/2 \). In contrast, for the piecewise defined profiles, the reflectivity oscillates as a function of thickness. Every piecewise defined profile had two attachment planes, at \( z_1 \) and \( z_2 \), in order to keep the interface symmetry and a bounded refractive index. The reflectivity oscillations in these cases were in accordance with thin film interference, with a thin film thickness of \( z_2 - z_1 \). Evidently discontinuities in the profile derivatives were causing reflections. However, it was not clear, only from the interference data, which type of phase change was the reflected wave undergoing at the \( z_1 \) and \( z_2 \) boundaries. We could only infer the relative phase between the two reflections. Our task now is to evaluate the phase change upon reflection, based on the interpretation of the amplitude equation numerical solutions.

A good example of how the junctions of a continuous but piecewise \( n(z) \) function still generate reflection, even when the interface is gradual, is shown in Fig. 2. The bottom graph displays reflectivity \( R \) versus interface thickness \( D \), in wavelength units, for two \( n(z) \) profile types. Both profiles are continuous but one of them, labeled as “tanh,” is analytic and the other, “linear,” is
interface, the one attachment point. To rule out reflections from the rest of the behaves, piecewise refractive index profiles can be built with only derivatives, but not in the third. To simplify the solutions inter-

Table 1 Refractive index profile classification.

<table>
<thead>
<tr>
<th>Profile type</th>
<th>( n(z) )</th>
<th>( \frac{dn}{dz} )</th>
<th>( \frac{d^2n}{dz^2} )</th>
<th>( \frac{d^3n}{dz^3} )</th>
<th>( \frac{d^4n}{dz^4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C^{-1} )</td>
<td>discontinuous</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C^0 )</td>
<td>continuous</td>
<td>discontinuous</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C^1 )</td>
<td>continuous</td>
<td>discontinuous</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C^2 )</td>
<td>continuous</td>
<td>discontinuous</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C^3 )</td>
<td>continuous</td>
<td>discontinuous</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

piecwise defined \[13\]:

\[
n_{\text{tanh}}(z) = \frac{n_1 + n_2}{2} + \frac{\Delta n}{2} \tanh \left( \frac{2.94}{D - z} \right),
\]

(13)

\[
n_{\text{linear}}(z) = \begin{cases} n_1 & z \leq \frac{-D}{1.8} \\ \frac{n_1 + n_2}{2} + \frac{\Delta n}{2} \left( 1 + \frac{z}{D/1.8} \right) & \frac{-D}{1.8} \leq z \leq \frac{D}{1.8} \\ n_2 & z \geq \frac{D}{1.8} \end{cases}
\]

(14)

where \( n_1 = 1 \) and \( n_2 = 1.5 \).

The reflectivity for the analytical profile “tanh” is almost null for \( D > 1.5z \). Nonetheless, for the piecewise profile “linear” \( R \) still oscillates due to interference of reflected waves at the junctions: \( z_1 = -D/1.8 \) and \( z_2 = D/1.8 \).

2. Refractive index profiles

In order to see how the reflectivity of a single junction profile behaves, piecewise refractive index profiles can be built with only one attachment point. To rule out reflections from the rest of the interface, the \( D \) parameter must be greater than \( 1.5z \). Let us classify the profiles \( n(z) \) according to the lowest order discontinuous derivative at the junction, as it is done for parametric continuity of curves \[21\]. This classification is shown in Table 1. For example, a \( C^2 \) profile is continuous in its first and second derivatives, but not in the third. To simplify the solutions interpretation, the junction is always placed at \( z = 0 \).

3. An abrupt interface

A good example of a \( C^{-1} \) type profile is the simple case of a hard interface between two homogeneous media, that can also be handled with the Fresnel formulae. We already know the outcome, so this is a good opportunity to check our results when using the amplitude equation \[8\]. Let us again resemble an air-glass barrier by choosing the corresponding refractive indices \( n_a = 1 \) and \( n_g = 1.5 \). For this profile type, \( D \) must be close to zero, this poses no problem since the barrier itself is the only source for reflection. Two profiles can be proposed:

\[
n_{\text{step}+}(z) = \begin{cases} n_a & z < 0 \\ n_g & z \geq 0 \end{cases}
\]

(15)

\[
n_{\text{step}−}(z) = \begin{cases} n_g & z < 0 \\ n_a & z \geq 0 \end{cases}
\]

(16)

Function \( n_{\text{step}+}(z) \) is increasing and \( n_{\text{step}−}(z) \) is decreasing. For all profiles, incident light comes from the left, along the \( z \) axis. In “step+” light passes from a lower refractive index medium to one of a higher index, while in “step−” the opposite occurs. The tags “+” and “−” in the function label are used to indicate if the function \( n(z) \) is increasing or decreasing.

The numerical solutions of the corresponding amplitude equations \[8\] are shown in Fig. 3. The solutions are plotted against the optical path in wavelength units \( A = \int_0^z n dz \) instead of \( z \), to ease interpretation further. Applying Eq. \[11\] for \( z < 0 \), reflectivities are found to be \( R_{\text{step}+} = 4.0000513% \) and \( R_{\text{step}−} = 4.0001383% \). The exact known value predicted by Fresnel formulae is

\[ R = \left( \frac{n_a - n_g}{n_a + n_g} \right)^2 \times 100 = 4% \]

the digits presented in the results confirm the accuracy of the numerical method used to solve Eq. \[8\]. Relative error is less than 0.0013% and deviation is \( \pm 0.00002\% \).

The local extrema of the oscillations in both graphs on Fig. 3 show where the incident and reflected waves are in or out of phase. Wherever there is a minimum, the electric vector fields of
Fig. 3. Amplitude equation solutions for (a) “step+”, $n_i = 1, n_o = 1.5$ and (b) “step−”, $n_i = 1.5, n_o = 1$. The junction is at $z = 0$.

Fig. 4. (a) Plot of the profile in Eq. (17) “lin&tanh”, it is constructed by joining a constant function with a hyperbolic tangent. It models a gradual interface with one discontinuity in the first derivative at $z = 0$, where this derivative is increasing. The amplitude equation solution $A_e$ for “lin&tanh+” profile is also shown and (b) a closer look at the solution in the oscillating region.

these waves point in opposite directions; if there is a maximum, it means the fields point in the same direction. If maxima occur at $A_{\text{max}} = (-1/2 - m/2)\lambda$ and minima at $A_{\text{min}} = (-m/2)\lambda$, for $m = 0, 1, 2, 3 \ldots$, the electric field waves must have phase difference of $\delta = \pi$ between them at $z = 0$ (the discontinuity site), thus revealing a phase change of $\pi$ due to reflection at the junction plane. If maxima and minima positions are interchanged, waves must be in phase at $z = 0$, revealing no phase change due to reflection at the junction plane. For the former interpretation to be valid, the refractive index $n(z)$ must be reasonably constant along the interval where the local extrema are found.

The location of the critical points for the “step+” case, shows an approximate phase change of $\pi$ upon reflection at $z = 0$. There is a mean offset of $(0.004)\pi$ from the exact $\delta = \pi$ value with very small deviation ($\sigma < \pi/1000$). For “step−”, maxima fall close to $A_{\text{max}} = (-m/2)\lambda$ and minima at $A_{\text{min}} = (-1/2 - m/2)\lambda$, revealing an almost null phase change due to reflection at the junction. However, there is a mean offset of $(0.004)\pi$ from the exact $\delta = 0$ value with standard deviation of $\sigma = (0.002)\pi$. These results match very well with Fresnel predictions.

4. Gradual and continuous interfaces

4.1. Type C0 profiles

A type C0 profile with only one piecewise junction can be constructed by joining a constant function with a hyperbolic tangent at $z = 0$:

$$n_{\text{lin&tanh+}}(z) = \begin{cases} 
n_i & z < 0 \\
n_o + (n_i - n_o) \tanh\left(\frac{z}{\lambda}\right) & z \geq 0,
\end{cases}$$ (17)

Here

$$a_1 = \frac{1}{2} \left( \ln\left(\frac{1.95}{0.05}\right) - \ln\left(\frac{1.05}{0.95}\right) \right),$$

so that $D$ is the interface thickness, the distance needed for $n$ to change 90% of the total $\Delta n = n_i - n_o$. The function $n_{\text{lin&tanh-}}(z)$ is continuous, at the junction it is increasing while its derivative $dn/dz$ is discontinuous and is also increasing. Interface thickness is chosen to be reasonably soft $D = 2\lambda$ to discard any significant source of reflection except at the junction. Fig. 4(a) shows a plot of this profile. Light is always assumed to be incident from the left. The numerical solution of the corresponding amplitude equation (8) is also plotted in Fig. 4.

In the case of “lin&tanh+” profile, applying Eq. (11) for $z < 0$, the reflectivity is found to be $R_{\text{lin&tanh+}} = 3.15565 \times 10^{-25}$, with a standard deviation of $\sigma_R = 1.8 \times 10^{-10}$. This reflectivity is much smaller than the abrupt interface result 4%, since it is related to a very gradual interface ($D = 2\lambda$). Yet, it is still far greater than the reflectivity computed for the analytic “tanh” profile stated in Eq. (13), with the same interface thickness, $R_{\text{tanh}} = 3.16840 \times 10^{-10}$. The first derivative discontinuity in $n(z)$, thus enhances the reflectivity substantially.

To analyze phase relationship between incident and reflected waves let us take as reference the reflection plane (always drawn at $z = 0$ in the figures). In this case, it lies where the first order derivative is discontinuous, although the refractive index itself is continuous. The phase between incident and reflected waves may be evaluated at any such region with constant refractive index. However, it is easier to consider a plane located at a multiple of the half-integer wavelength, $A = -m(\lambda/2)$ where $m$ is an integer. From any of these planes, the incident wave travels to the reflection plane and then the reflected part travels back an integer

$$\begin{align*}
\text{Field amplitude } A_e & = \frac{1}{2} \ln\left(\frac{1.95}{0.05}\right) - \ln\left(\frac{1.05}{0.95}\right), \\
\text{Field amplitude } A_e & = \frac{1}{2} \ln\left(\frac{1.95}{0.05}\right) - \ln\left(\frac{1.05}{0.95}\right). \\
\text{Field amplitude } A_e & = \frac{1}{2} \ln\left(\frac{1.95}{0.05}\right) - \ln\left(\frac{1.05}{0.95}\right).
\end{align*}$$
number of wavelengths, and thus no phase needs to be added due to refraction at the derivative discontinuity \((\lambda = 0)\), the reflected wave must be in phase with the incident wave after \(m\) wavelengths travel. However, consider the plane \(A = -1.0\lambda\) in Fig. 4(b). The incident and reflected waves are not in phase. In the middle between one amplitude maximum and a contiguous minimum, these waves must have a relative phase difference of \(\delta = (\pi/2)\).

The numerical solution shown at \(A = -1.0\lambda\) in Fig. 4(b), reveals a phase difference of \(\pi/2\). Therefore, there must be a phase change of \(\delta = \pi/2\) introduced by the reflection at the plane where the derivative is discontinuous. A detailed analysis of the numerical solution shows that there is only a small offset of \((-0.024 \pm 0.004)\pi/2\) from the exact \(\delta = \pi/2\) value. This is certainly a very surprising result, since we know from Fresnel equations, that a phase change of \(\pi\) is introduced between two transparent dielectrics whose refractive index is equal at the interface but its first derivative is discontinuous. The former “lin\&tanh” profile is increasing at the junction plane as well as the derivative \(dn/dz\), to see what would be the case for an increasing profile \(n(t)\) but decreasing first derivative \(dn/dz\) let us devise another \(C^0\) type profile. Let us again piecewise join a hyperbolic tangent with a constant function at \(z=0\), but this time placing the hyperbolic tangent first:

\[
R_{\text{lin\&tanh}^+}(z) = \begin{cases} 
\frac{n_0 + (n_a - n_0) \tanh(z/L)}{n_0} & z < 0 \\
\frac{n_0 + (n_a - n_0) \tanh(z/L)}{n_a} & z > 0.
\end{cases}
\]  

Once more, the interface thickness is \(D = 2\lambda\) to discard any significant source of reflection except for the junction.

**Fig. 5.** (a) Plot of the profile in Eq. (18) “tanh\&lin”, it is constructed by joining a hyperbolic tangent with a constant function. It models a gradual interface with \(D = 2\lambda\) and one discontinuity in the first derivative at \(z=0\), where this derivative is decreasing. The amplitude equation solution \(A_z\) for “tanh\&lin” profile is also shown and (b) a closer look at the solution in the oscillating region.

**Table 2**
Calculated reflectivity and approximate phase difference \(\delta\) between incident and reflected waves at the reflection plane, \(z=0\), related to the type \(C^0\) profiles. The symbol \(\sigma_p\) stands for the reflectivity standard deviation.

<table>
<thead>
<tr>
<th>(C^0) function</th>
<th>Reflectivity (%) (R)</th>
<th>Reflectivity relative standard deviation (%) (\sigma_p/R\times100)</th>
<th>Approximate phase change at reflection plane (z=0)</th>
<th>Offset from the exact (\pi/2) or (\pi/2) value in (\pi) units</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin&amp;tanh+</td>
<td>3.15565 \times 10^{-2}</td>
<td>0.0057</td>
<td>(\pi/2)</td>
<td>(-0.024 \pm 0.003)</td>
</tr>
<tr>
<td>tanh&amp;lin+</td>
<td>6.27498 \times 10^{-3}</td>
<td>1.1</td>
<td>(\pi/2)</td>
<td>(-0.014 \pm 0.003)</td>
</tr>
<tr>
<td>lin&amp;tanh−</td>
<td>6.24063 \times 10^{-3}</td>
<td>0.013</td>
<td>(\pi/2)</td>
<td>(0.005 \pm 0.006)</td>
</tr>
<tr>
<td>tanh&amp;lin−</td>
<td>3.16123 \times 10^{-2}</td>
<td>0.48</td>
<td>(\pi/2)</td>
<td>(0.019 \pm 0.005)</td>
</tr>
</tbody>
</table>
amplitude equation solutions we arrive at similar outcomes as abridged for all four $C^0$ profiles in Table 2.

4.1.1. Type $C^1$ profiles

To build some $C^1$ type profiles, in a similar fashion as with the $C^0$ type, a constant function is piecewise joined with a squared hyperbolic secant at its critical point in four different ways, shown in Table 3. Plots of the numerical solutions for the amplitude equation (8), taking $D = 2\lambda$, are illustrated in Fig. 6.

Reflectivity for type $C^1$ profiles, estimated with Eq. (11), and phase difference between incident and reflected waves at the reflection plane are reported in Table 4. The reflectivity is even lower for these profiles. Incident and reflected waves are in phase at the junction.

Table 3

<table>
<thead>
<tr>
<th>Function label</th>
<th>$n(z)$</th>
<th>$\frac{d^2 n}{dz^2}$ at the junction</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin&amp;sech +</td>
<td>$n_z$</td>
<td>Increasing</td>
</tr>
<tr>
<td></td>
<td>$n_z - (n_z - n_a) \text{ sech}^2 (\frac{n_a}{D})$</td>
<td>Decreasing</td>
</tr>
<tr>
<td>sech&amp;lin +</td>
<td>$n_z + (n_z - n_a) \text{ sech}^2 (\frac{n_a}{D})$</td>
<td>Increasing</td>
</tr>
<tr>
<td>lin&amp;sech –</td>
<td>$n_z$</td>
<td>Decreasing</td>
</tr>
<tr>
<td>sech&amp;lin –</td>
<td>$n_z - (n_z - n_a) \text{ sech}^2 (\frac{n_a}{D})$</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>$C^1$ function</th>
<th>Reflectivity ($%$) $R$</th>
<th>Reflectivity relative standard deviation ($%$) $\frac{\sigma_R}{R} \times 100$</th>
<th>Approximate phase change at reflection plane $z=0$</th>
<th>Offset from the exact $\pi$ or 0 value in $\pi$ units</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin&amp;sech +</td>
<td>1.15117 $\times 10^{-3}$</td>
<td>0.0052</td>
<td>0</td>
<td>$-0.004 \pm 0.003$</td>
</tr>
<tr>
<td>sech&amp;lin +</td>
<td>8.15438 $\times 10^{-5}$</td>
<td>1.4</td>
<td>0</td>
<td>$0.002 \pm 0.003$</td>
</tr>
<tr>
<td>lin&amp;sech –</td>
<td>8.10566 $\times 10^{-5}$</td>
<td>0.0010</td>
<td>$\pi$</td>
<td>$0.002 \pm 0.005$</td>
</tr>
<tr>
<td>sech&amp;lin –</td>
<td>1.15005 $\times 10^{-1}$</td>
<td>0.36</td>
<td>$\pi$</td>
<td>$0.006 \pm 0.005$</td>
</tr>
</tbody>
</table>

$\delta = 0$, for profiles with an increasing $\frac{d^2 n}{dz^2}$ at the junction, in accordance with the critical point positions. They are out of phase $\delta = \pi$ for profiles with a decreasing $\frac{d^2 n}{dz^2}$ at the junction.

4.1.2. Type $C^2$ profiles

Four $C^2$ type profiles are built by piecewise joining a constant function with a cubic exponential at its critical and inflection point, as shown in Table 5. Plots of numerical solutions for the amplitude equation (8), taking $D = 2\lambda$, are shown in Fig. 7.

Reflectivity for $C^2$ type profiles, estimated with Eq. (11), and phase difference between incident and reflected waves at the reflection plane are reported in Table 6. Reflectivity drops more than one order of magnitude for these profiles with respect to the

Table 5

<table>
<thead>
<tr>
<th>Label</th>
<th>$n(z)$</th>
<th>$\frac{d^2 n}{dz^2}$ at the junction</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin&amp;cubexp +</td>
<td>$n_a$</td>
<td>Increasing</td>
</tr>
<tr>
<td></td>
<td>$n_a - (n_a - n_b) \exp\left(-\frac{n_b}{D}\right)^3$</td>
<td>Decreasing</td>
</tr>
<tr>
<td>lin&amp;cubexp –</td>
<td>$n_a$</td>
<td>Decreasing</td>
</tr>
<tr>
<td></td>
<td>$n_a - (n_a - n_b) \exp\left(-\frac{n_b}{D}\right)^3$</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

Fig. 6. Amplitude equation solutions for type $C^1$ profiles, a close look at the solutions in the oscillating regions. (a) Solutions for “lin&sech +” and “sech&lin +” and (b) solutions for “lin&sech –” and “sech&lin –”.
(b) solutions for "lin&cubexp" and reflected waves have phase difference for profiles with an increasing inflection point, as shown in Table 7. Plots of the numerical constant function with a quartic exponential at its critical and previous phase upon reflection will not be so close to zero or pi.

Table 6
Calculated reflectivity and approximate phase difference \( \delta \) between incident and reflected waves at the reflection plane, \( z=0 \), related to the type \( C^2 \) profiles. The symbol \( \sigma_s \) stands for the reflectivity standard deviation.

<table>
<thead>
<tr>
<th>( C^2 ) function</th>
<th>Reflectivity (%) ( R )</th>
<th>Reflectivity relative standard deviation (%) ( \sigma_s / R \times 100 )</th>
<th>Approximate phase change at reflection plane ( z=0 )</th>
<th>Offset from the exact ( \pi / 2 ) or ( \pi / 2 ) value in ( \pi ) units</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin&amp;cubexp+</td>
<td>1.33978 \times 10^{-6}</td>
<td>0.0045</td>
<td>( \pi / 2 )</td>
<td>( -0.001 \pm 0.003 )</td>
</tr>
<tr>
<td>cubexp&amp;lin+</td>
<td>5.15290 \times 10^{-6}</td>
<td>1.2</td>
<td>( \pi / 2 )</td>
<td>0.004 \pm 0.004</td>
</tr>
<tr>
<td>lin&amp;cubexp−</td>
<td>5.18475 \times 10^{-6}</td>
<td>0.020</td>
<td>( \pi / 2 )</td>
<td>0.015 \pm 0.005</td>
</tr>
<tr>
<td>cubexp&amp;lin−</td>
<td>1.33984 \times 10^{-6}</td>
<td>0.014</td>
<td>( \pi / 2 )</td>
<td>0.011 \pm 0.005</td>
</tr>
</tbody>
</table>

Table 7
Type \( C^3 \) profiles. The piecewise junction is always at \( z=0 \). Here \( a_z = \sqrt{\text{ln}(0.95)} - \sqrt{\text{ln}(0.05)} \) so that \( D \) is the interface thickness, the distance needed for \( n \) to change 90% of the total \( \Delta n = n_2 - n_3 \).

<table>
<thead>
<tr>
<th>Label</th>
<th>( n(z) )</th>
<th>( d^4n / dz^4 ) at the junction</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin&amp;quartexp+</td>
<td>( n_x )</td>
<td>Increasing ( n_x - (n_2 - n_3) \exp\left(-\frac{a_z}{D^2}\right)^4 ) at ( z &lt; 0 )</td>
</tr>
<tr>
<td>quartexp&amp;lin+</td>
<td>( n_x + (n_2 - n_3) \exp\left(-\frac{a_z}{D^2}\right)^4 )</td>
<td>Increasing ( n_x ) at ( z \geq 0 )</td>
</tr>
<tr>
<td>lin&amp;quartexp−</td>
<td>( n_x )</td>
<td>Decreasing ( n_x + (n_2 - n_3) \exp\left(-\frac{a_z}{D^2}\right)^4 ) at ( z &lt; 0 )</td>
</tr>
<tr>
<td>quartexp&amp;lin−</td>
<td>( n_x - (n_2 - n_3) \exp\left(-\frac{a_z}{D^2}\right)^4 )</td>
<td>Decreasing ( n_x ) at ( z \geq 0 )</td>
</tr>
</tbody>
</table>

previous \( C^1 \) type profiles. According to the critical point positions, for profiles with an increasing \( d^4n / dz^4 \) at the junction, incident and reflected waves have phase difference \( \delta = -\pi / 2 \), while for those with a decreasing \( d^4n / dz^4 \) at the junction \( \delta = \pi / 2 \).

4.1.3. Type \( C^3 \) profiles
Finally, four \( C^3 \) type profiles are built by piecewise joining a constant function with a quartic exponential at its critical and inflection point, as shown in Table 7. Plots of the numerical solutions for the amplitude equation (8), taking \( D = 2\lambda \), are shown in Fig. 8.

Reflectivity for type \( C^3 \) profiles, estimated with Eq. (11), and phase difference between incident and reflected waves at the reflection plane are reported in Table 8. The numerical algorithm yields a null deviation for the reflectivity in these cases due to numerical round-off, since we are now close to the accuracy limit of this method. Reflectivity is even lower for these \( C^3 \) profiles in comparison with all the previous profile types. From the critical point positions, profiles with an increasing \( d^4n / dz^4 \) at the junction, have a phase difference between incident and reflected waves close to \( \delta = \pi \). For those with a decreasing \( d^4n / dz^4 \) at the junction, these counter propagating waves are almost in phase \( \delta = 0 \).

For “quartexp&lin+” and “lin&quartexp−” a greater departure from the exact \( \pi \) or 0 values is obtained. Recall that the reflectivity produced by the analytic part of the profile must be close to the one produced by a fully analytic profile, for example “tanh” reflectivity is \( R_{\text{tanh}} = 3.16840 \times 10^{-10} \), Therefore, the reflectivities coming from these two contributions are similar for these profiles, with \( D = 2\lambda \). However, for the analytic part of \( n(z) \), the reflectivity plane is smeared out. Hence, the overall phase upon reflection will not be so close to zero or pi.

5. Conclusions
The reflectivity diminishes as the lowest order of the discontinuous derivatives is greater, even if the interface thickness \( D \) remains constant. The selected profiles and their amplitude equation solutions show a certain order, related to the phase difference between the incident and reflected waves at the
selected profiles for each type constitute two reversibility couples. Reflectivity is expected to be the same for both members each couple [22,23]. Our numeric results follow this expectation within the deviation interval ±σR. For all these reversibility couples, σR is smaller when light is incident from the homogeneous side, where n(z) is strictly constant within the −∞ < z < 0 interval. Standard deviation σR is greater in the other case, when light is incident from the inhomogeneous side, because reflectivity it is calculated via Eq. (11) which is valid for a homogeneous medium. The oscillating region of the solution Aq falls on the slowly but still varying refractive index side.

Results in Table 9 suggest a more general conjecture: For a Cm profile type (lowest order discontinuous derivative d(m−1)n/dz(m−1), phase change upon reflection at the discontinuity plane is (1−m)π/2 for an increasing lowest order discontinuous derivative. Phase change upon reflection is (−1−m)π/2 for a decreasing lowest order discontinuous derivative.

By modeling light propagation with the amplitude and phase representation, we have presented how discontinuities in the refractive index derivatives enhance reflection in a transparent stratified medium. We have shown too that depending on the order of the discontinuity, the phase change upon reflection obtained from the numerical results can be ±(π/2) in addition to the usual 0, or ±π obtained for dielectric abrupt interfaces at normal incidence. These results may be applied to rugate filter design, interpretation of Doppler radar measurements in the clear air atmosphere, optical coherence tomography and other topics related with wave propagation in stratified media.

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References


