Dielectric interfaces and mirrors in the amplitude and phase representation

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Reflection and refraction at an abrupt dielectric interface at normal incidence is evaluated using an amplitude and phase (amph) formalism. The optical properties of a stack with two quarter-wavelength layers is then calculated. The characteristics of a mirror constructed with quarter-wavelength layers are discussed using the amplitude and phase representation. Floquet nonlinear theorem is invoked to describe the multi layered system. Results are consistent with Fresnel formulae and conventional matrix methods for stratified periodic media. However, the amph formalism offers several advantages: i) it is capable of showing the field properties as it propagates through the stack, ii) it gives a lucid physical insight because the variables involved have a clear physical meaning, iii) the mathematical description is simple.

Keywords: wave propagation; stratified media; wave invariants

1. Introduction

The amplitude and phase of light upon reflection and refraction at an abrupt plane interface is described by Fresnel equations. When the setup involves two or more parallel interfaces, multiple beam interference comes into play. The Fabry-Perot etalon with either fixed or variable distance between mirrors (scanning) is a widely spectroscopic technique. A higher number of interfaces makes the problem much richer but also a lot more complicated. Alternate periodic homogeneous layers are mostly tackled with a matrix formalism [1]. One of the most important advantages of the matrix formalism is its ability to deal with periodic profiles. Powers of the one period matrix model the periodic material. Dielectric mirror and anti-reflection coatings are have been designed using these tools in the last decades [2, 3]. However, these methods become rather complicated when modeling rugate films or photonic crystals, that is, periodic or non periodic layers where the layers are not homogeneous [4, 5]. More complex stratified media pose even more challenging modeling problems. Photonic crystals design has extended the matrix formalism to other potentials [6] and has also stimulated other theoretical approaches such as the semiclassical coupled-wave theory [7] and the Floquet-Bloch wave description [8].

In the present formalism, the fields are described in terms of the amplitude and phase variables concomitant to wave phenomena. Numerical solutions to fairly complicated profiles have been found using this method [9]. Localized reflectivity enhancement as well as atypical phase changes due to discontinuities in the derivatives of the refractive index have been predicted both numerically [10] and analytically in a slowly varying refractive index approximation [11]. This communication is devoted to reproduce some exact analytical results. Three simple cases will be presented to expound the philosophy of the method. Namely, an abrupt dielectric interface at normal incidence, a $\lambda/4$
homogeneous layer and a stack of alternating pairs of $\lambda/4$ layers with different refractive indices. In section 2, the elementary theory is presented. Subsection 2.2 is of special importance because it introduces the invariant that, on the one hand, decouples the amplitude and phase differential equations and on the other, plays a key role for evaluating the propagation of the field. The abrupt interface is considered in section 3, while a $\lambda/4$ homogeneous layer is studied in section 4. A $\lambda/4$ periodic medium is described in section 5. A nonlinear version of Floquet theorem is invoked to write the solution in terms of the periodicity. Conclusions are drawn in the last section.

2. Preamble

2.1. Counter-propagating waves

In one dimension, it is possible to have a running wave or counter-propagating waves with different amplitudes. In the degenerate case when the counter-propagating waves have equal amplitude, they are often called 'stationary' or 'standing'. These terms should be avoided for, although the nodes are spatially localized, the waves propagate unhindered. The most general 1D description is given in terms of counter-propagating waves. Consider the complex representation of two scalar waves

$$ae^{i\alpha} = a \cos \alpha + ia \sin \alpha, \quad be^{i\beta} = b \cos \beta + ib \sin \beta.$$ 

Their sum is

$$ae^{i\alpha} + be^{i\beta} = (a \cos \alpha + b \cos \beta) + i (a \sin \alpha + b \sin \beta).$$

The polar representation of the sum is

$$\rho e^{i\phi} = \sqrt{(a \cos \alpha + b \cos \beta)^2 + (a \sin \alpha + b \sin \beta)^2} \exp \left[ i \arctan \left( \frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta} \right) \right],$$

but the squares in the radicand are equal to

$$(a \cos \alpha + b \cos \beta)^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + 2ab \cos \alpha \cos \beta,$$

$$(a \sin \alpha + b \sin \beta)^2 = a^2 \sin^2 \alpha + b^2 \sin^2 \beta + 2ab \sin \alpha \sin \beta,$$

thus, the squared amplitude is $a^2 + b^2 + 2ab \cos (\alpha - \beta)$. The polar representation of the sum is then

$$\rho e^{i\phi} = \sqrt{a^2 + b^2 + 2ab \cos (\alpha - \beta)} \exp \left[ i \arctan \left( \frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta} \right) \right]. \quad (1)$$

If the waves are counter-propagating, their phases are equal but opposite, thus $\beta = -\alpha$

$$\rho_{++} e^{i\phi_{++}} = \sqrt{a^2 + b^2 + 2ab \cos (2\alpha)} \exp \left[ i \arctan \left( \frac{a - b}{a + b} \tan \alpha \right) \right]. \quad (2)$$

2.1.1. Quotient of counter-propagating sum and difference amplitudes

The double angle trigonometric identity $\cos (2\alpha) = 2 \cos^2 \alpha - 1$, allows the square amplitude to be rearranged as $a^2 + b^2 + 2ab \cos (2\alpha) = (a - b)^2 + 4ab \cos^2 \alpha$. The $4ab$ factor can be written in terms of the sum and difference of amplitudes as $4ab = (a + b)^2 - (a - b)^2$. The square amplitude can then
be written as
\[
a^2 + b^2 + 2ab \cos (2\alpha) = (a - b)^2 + \left( (a + b)^2 - (a - b)^2 \right) \cos^2 \alpha
\]
\[
= (a - b)^2 \left\{ 1 + \frac{(a + b)^2 - 1}{(a - b)^2} \cos^2 \alpha \right\}.
\]

Define the variable \( \mu \) as the quotient of the sum over the difference of the counter-propagating amplitudes
\[
\mu = \frac{a + b}{a - b}.
\]
The amplitude is then
\[
\rho_{\leftrightarrow} = \frac{1}{\mu} (a + b) \sqrt{1 + (\mu^2 - 1) \cos^2 \alpha}, \text{ but } \mu = \frac{a + b}{a - b} = \frac{(a + b)^2}{a^2 - b^2}, \text{ thus } a + b = \sqrt{\mu} \sqrt{a^2 - b^2}, \text{ and the amplitude can be written as}
\]
\[
\rho_{\leftrightarrow} = \frac{1}{\sqrt{\mu}} \sqrt{a^2 - b^2} \sqrt{1 + (\mu^2 - 1) \cos^2 \alpha}.
\]

If the wave \( be^{-i\alpha} \) arises from the reflection of the wave \( ae^{i\alpha} \), the reflection coefficient is \( r = b/a \). The \( \mu \) quotient in terms of the reflectivity is
\[
\mu = \frac{a + b}{a - b} = \frac{1 + r}{1 - r},
\]
whereas the reflectivity in terms of the \( \mu \) quotient is
\[
r = \frac{b}{a} = \frac{\mu - 1}{\mu + 1}.
\]
The polar representation of the counter propagating waves in terms of \( \mu \) is
\[
\rho_{\leftrightarrow} e^{i\phi_{\leftrightarrow}} = \frac{1}{\sqrt{\mu}} \sqrt{a^2 - b^2} \sqrt{1 + (\mu^2 - 1) \cos^2 \alpha} \exp \left[ i \arctan \left( \frac{1}{\mu} \tan \alpha \right) \right].
\]

### 2.2. Energy exchange invariant

The two linearly independent solutions of the wave equation can be interpreted as two complementary fields [12]. Oscillations arise from the dynamic equilibrium or imbalance between these two fields. A very general conservation equation of the form \( \nabla \cdot (\nabla \psi) + \frac{\partial}{\partial t} \psi = 0 \) can be stated in terms of the two complementary fields. The density and flow that originate from this continuity equation are
\[
\psi \equiv \left( \psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t} \right), \quad \nabla \psi \equiv (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2).
\]

In terms of amplitude \( \rho \) and phase \( \phi \) variables, the density and flow are
\[
\psi = \rho^2 (r) \frac{\partial}{\partial t} \phi (r), \quad \nabla \psi = \rho^2 (r) \nabla \phi (r).
\]

This expression for the flow, except for a factor of \( 1/\omega \), is equal to Poynting's vector evaluated for a linearly polarized monochromatic wave in an inhomogeneous medium. Since \( \omega \) is constant for
a monochromatic wave, these two results are equivalent. The complementary fields flow may then be interpreted as the energy flow times the number of cycles. However, this interpretation is not correct if the wave amplitude and frequency are time dependent because then, the complementary fields procedure or Poynting’s theorem do not lead to the same conservation equation [13]. For a monochromatic scalar field in one dimension, the flow $\triangleright \psi$ becomes an invariant

$$\triangleright \psi = Q = \rho^2(z) \frac{\partial \phi(z)}{\partial z}. \quad (10)$$

This complementary fields or energy exchange invariant plays a fundamental role in the propagation of the wave through a stack of layers or a continuously stratified medium. Such an exact invariant also arises in mechanical problems involving the time dependent harmonic oscillator [14]. It is often referred in the literature as the Ermakov or the Ermakov-Pinney invariant. The nonlinear amplitude equation derived from the 1+1 dimensional wave equation for monochromatic waves is [15]

$$\rho^3 \frac{d^2 \rho}{dz^2} + \rho^4 \Omega^2 = Q^2, \quad (11)$$

where $\Omega^2$ is the spatially dependent parameter that models the inhomogeneous medium. The spatial frequency defined as the derivative of the phase function $\frac{\partial \phi(z)}{\partial z} \equiv \kappa$, fulfills the nonlinear differential equation

$$\kappa \frac{d^2 \kappa}{dz^2} - \frac{3}{2} \left( \frac{d\kappa}{dz} \right)^2 + 2 \left[ \kappa^2 - \Omega^2 \right] \kappa^2 = 0. \quad (12)$$

These two differential equations request that the amplitude $\rho$ is at least a class $C^2$ function and the phase $\phi$ is at least class $C^3$. These requirements impose continuity conditions when the solutions are piece-wise proposed.

2.2.1. Counter-propagating waves invariant

In order to assess the energy exchange flow for two counter-propagating waves with constant amplitudes $a, b$ and opposite linear phase $\alpha = kz$, evaluate the derivative of the counter-propagating phase $\phi_{\leftrightarrow}$

$$\frac{d\phi_{\leftrightarrow}}{dz} = \frac{d}{dz} \left[ \arctan \left( \frac{1}{\mu} \tan kz \right) \right] = \frac{1}{\mu} k \sec^2 kz = \frac{1}{\mu} \frac{k}{\left( \frac{1}{\mu} \right)^2 \tan^2 kz + 1} = \frac{1}{\mu} \frac{k}{\left( \frac{1}{\mu} \right)^2 \sin^2 kz + \cos^2 kz}. \quad (13)$$

The $\mu$ quotient and the wave vector magnitude $k$ are constants in homogeneous regions of the medium. Write this expression solely in terms of the cosine function and multiply and divide by $\mu^2$ to obtain

$$\frac{d\phi_{\leftrightarrow}}{dz} = \mu k \left( 1 - \cos^2 kz \right) + \mu^2 \cos^2 kz = \frac{\mu k}{1 + (\mu^2 - 1) \cos^2 kz}. \quad (13)$$

Recall the expression for the counter-propagating amplitude $\rho_{\leftrightarrow}$ written in terms of the $\mu$ variable (4). The energy exchange invariant (10) is then

$$Q = \rho^2_{\leftrightarrow} \frac{d\phi_{\leftrightarrow}}{dz} = \frac{1}{\mu} \left( a^2 - b^2 \right) \left( 1 + (\mu^2 - 1) \cos^2 kz \right) \frac{\mu k}{1 + (\mu^2 - 1) \cos^2 kz}.$$
thus

\[ Q = \rho_2^2 \frac{d\varphi_{\leftrightarrow}}{dz} = \left( a^2 - b^2 \right) k. \]  \hspace{1cm} (14)

The energy exchange flow invariant for two counter-propagating waves in regions where the refractive index is constant of an otherwise inhomogeneous medium is then proportional to the square amplitude difference and is linearly dependent with the wave vector magnitude.

3. Abrupt interface

Consider two semi-infinite media 1 and 2 with refractive indices \( n_1 \) and \( n_2 \) respectively that meet at the \( z_{12} \) plane. Propose amplitude solutions in each layer of the form

\[ \rho_{1\leftrightarrow} = \frac{1}{\sqrt{\mu_1}} \sqrt{a_1^2 - b_1^2} \sqrt{1 + (\mu_1^2 - 1) \cos^2 (k_1 z)} \]  \hspace{1cm} (15)

and

\[ \rho_{2\leftrightarrow} = \frac{1}{\sqrt{\mu_2}} \sqrt{a_2^2 - b_2^2} \sqrt{1 + (\mu_2^2 - 1) \cos^2 (k_2 z + \varphi_{12})}. \]  \hspace{1cm} (16)

The \( \mu \) quotients are

\[ \mu_j = \frac{a_j + b_j}{a_j - b_j}, \]  \hspace{1cm} (17)

for \( j = 1, 2 \); Whereas their phases are

\[ \varphi_{1\leftrightarrow} = \arctan \left( \frac{1}{\mu_1} \tan k_1 z \right) \]  \hspace{1cm} (18)

and

\[ \varphi_{2\leftrightarrow} = \arctan \left( \frac{1}{\mu_2} \tan (k_2 z + \varphi_{12}) \right) \]  \hspace{1cm} (19)

respectively.

3.1. Energy exchange flow conservation

From the invariant of counter-propagating waves (14), \( (a_1^2 - b_1^2) k_1 = (a_2^2 - b_2^2) k_2 \). Since \( k_j = n_j k_0 \), then \( (a_1^2 - b_1^2) n_1 = (a_2^2 - b_2^2) n_2 \). If there is no incoming wave in medium 1, \( b_1 = 0 \); The invariant exchange energy flow conservation is then

\[ a_1^2 = (a_2^2 - b_2^2) n, \]  \hspace{1cm} (20)

where \( n = \frac{n_2}{n_1} \) is the quotient of the refractive indices in the two media.
3.2. continuity

If the medium is abrupt, the two amplitude solutions should be equal at the interface plane $z_{12}$

$$\rho_{1\leftrightarrow} (z_{12}) = 1 / \sqrt{\mu_1} a_1 \sqrt{1 + (\mu_1^2 - 1) \cos^2 k_1z_{12}} = \rho_{2\leftrightarrow} (z_{12})$$

$$= 1 / \sqrt{\mu_2} \sqrt{a_2^2 - b_2^2 \sqrt{1 + (\mu_2^2 - 1) \cos^2 (k_2z_{12} + \varphi_{12})}}.$$

Notice that if $b_1 = 0$, then $\mu_1 = 1$ and the amplitude is constant; Hence the value of the phase in the square cosine for $\rho_{1\leftrightarrow}$ is irrelevant.

$$\rho_{1\leftrightarrow} (z_{12}) = a_1 = \rho_{2\leftrightarrow} (z_{12}) = 1 / \sqrt{\mu_2} \sqrt{a_2^2 - b_2^2 \left\{ \sqrt{1 + (\mu_2^2 - 1) \cos^2 (k_2z_{12} + \varphi_{12})} \right\}.$$

This result reveals consistency because it means that if the amplitude is not modulated, it is the same regardless of the position where it is evaluated. Let the origin be located at the $z_{12}$ plane, i.e. $z_{12} = 0$, the initial phase is thus zero. From (18), $\phi_{1\leftrightarrow} (z_{12}) = 0$, and since the phase solutions must be continuous at $z_{12}$, $\phi_{1\leftrightarrow} (z_{12}) = \phi_{2\leftrightarrow} (z_{12})$. Therefore, $k_2z_{12} + \varphi_{12} = 0 \Rightarrow \varphi_{12} = 0$. The amplitude continuity equation is then

$$a_1 = 1 / \sqrt{\mu_2} \sqrt{a_2^2 - b_2^2 \sqrt{1 + (\mu_2^2 - 1) \cos^2 (k_2z_{12} + \varphi_{12})}} = \sqrt{a_2^2 - b_2^2 \sqrt{\mu_2}}.$$

Invoking the invariant equation (20),

$$\mu_2 = \frac{a_1}{a_2^2 - b_2^2} = n. \quad (21)$$

The amplitude function in medium 2 is then

$$\rho_{2\leftrightarrow} (z) = \frac{a_1}{n} \sqrt{1 + (n^2 - 1) \cos^2 (k_2z)}, \quad (22)$$

The reflectivity, from (6) is

$$r_2 = \frac{b_2}{a_2} = \frac{\mu_2 - 1}{\mu_2 + 1}.$$  

Substitution of (21) gives the usual result for reflection between two dielectric media at normal incidence

$$r_2 = \frac{\mu_2 - 1}{\mu_2 + 1} = \frac{n_2 - 1}{n_2 + 1} = \frac{n_2 - n_1}{n_2 + n_1}.$$  

The reflected amplitude is $b_2 = r_2a_2$. From the invariant relationship

$$a_1^2 = (a_2^2 - b_2^2) n = (a_2^2 - r_2^2 a_2^2) n = a_2^2 \left( 1 - r_2^2 \right) n.$$  

If the incident amplitude $a_2$ is normalized to 1, $a_1^2 = (1 - r_2^2) n$, the transmitted amplitude is then

$$a_1 = \frac{2n}{(n + 1)}.$$
Figure 1. Medium with refractive index $n_1$ for $z < z_{12}$. A layer with refractive index $n_2$ is placed between $z_{12}$ and $z_{23}$ with $\lambda/4$ optical path. Thereafter the medium has refractive index $n_2$ in the non-periodic case (dotted line). In the periodic structure, it returns to $n_1$ and the optical path between $z_{23}$ and $z_{31}$ is also $\lambda/4$.

The amplitude function normalized to the incident wave $a_2 = 1$ is

$$\rho_{2\leftrightarrow}(z) = \frac{2}{n + 1} \sqrt{1 + (n^2 - 1) \cos^2(k_2 z)}.$$

4. Medium with $\lambda/4$ homogeneous layer

Consider now a material whose medium 1 with refractive index $n_1$ is again semi-infinite. It encounters medium $n_2$ at the $z_{12}$ plane as in the previous case, but the optical path of this medium is now $\lambda/4$, as shown in figure 1.

4.1. Second interface

Consider that there is a second interface at $z_{23}$ as shown in figure 1. Let the amplitude solution in region 3 with refractive index $n_3$, to be of the form of counter propagating waves

$$\rho_{3\leftrightarrow}(z) = \frac{1}{\sqrt{\mu_3}} \sqrt{a_3^2 - b_3^2} \left\{ \sqrt{1 + (\mu_3^2 - 1) \cos^2(k_3 z + \varphi_{23})} \right\}$$

and its phase

$$\phi_{3\leftrightarrow}(z) = \arctan \left[ \frac{1}{\mu_3} \tan (k_3 z + \varphi_{23}) \right].$$

From the invariant of counter-propagating waves (14),

$$a_1^2 n_1 = \left( a_2^2 - b_2^2 \right) n_2 = \left( a_3^2 - b_3^2 \right) n_3 \quad (23)$$

If the medium is abrupt, the two amplitude solutions should be equal at the interface

$$\rho_{2\leftrightarrow}(z_{23}) = \frac{1}{\sqrt{\mu_2}} \sqrt{a_2^2 - b_2^2} \sqrt{1 + (\mu_2^2 - 1) \cos^2(k_2 z_{23})} = \rho_{3\leftrightarrow}(z_{23})$$

$$= \frac{1}{\sqrt{\mu_3}} \sqrt{a_3^2 - b_3^2} \sqrt{1 + (\mu_3^2 - 1) \cos^2(k_3 z_{23} + \varphi_{23})}.$$
For a medium with optical path of $\frac{\lambda}{4}$, the phase is $k_2 z_{23} = k_0 n_2 z_{23} = \frac{\pi}{2}$. Since the counter-propagating phase at the interface must be equal $\phi_{2\leftrightarrow}(z_{23}) = \phi_{3\leftrightarrow}(z_{23})$,

$$(k_3 z_{23} + \varphi_{23}) = \arctan \left[ \frac{\mu_3}{\mu_2} \tan \frac{\pi}{2} \right] = \frac{\pi}{2}$$

regardless of the value of $\frac{\mu_3}{\mu_2}$. Thus,

$$\varphi_{23} = \frac{\pi}{2} - k_3 z_{23} = \frac{\pi}{2} - k_0 n_3 \left( \frac{\pi}{2} \frac{1}{k_0 n_2} \right) = \frac{\pi}{2} \left( 1 - \frac{n_3}{n_2} \right).$$

The amplitude continuity equality is then

$$\rho_{2\leftrightarrow}(z_{23}) = \frac{1}{\sqrt{\mu_2}} \sqrt{a_2^2 - b_2^2} = \rho_{3\leftrightarrow}(z_{23}) = \frac{1}{\sqrt{\mu_3}} \sqrt{a_3^2 - b_3^2},$$

thus, from the invariant relationship (23), if the third medium has the same refractive index of the first medium, $n_3 = n_1$,

$$\mu_3 = \mu_2 \frac{a_3^2 - b_3^2}{a_2^2 - b_2^2} = \frac{n_2}{n_1} n_2 = n_2^2.$$

The reflectivity of the quarter wave slab is then

$$r_3 = \frac{b_3}{a_3} = \frac{\mu_3 - 1}{\mu_3 + 1} = \frac{n_2^2 - 1}{n_2^2 + 1}.$$

The reflected amplitude is $b_3 = r_2 a_3$. From the invariant relationship

$$a_1^2 = \left( a_3^2 - b_3^2 \right) = a_3^2 \left( 1 - r_3^2 \right).$$

For unit incident amplitude, the transmitted amplitude $a_1^2 = (1 - r_3^2)$ is

$$a_1 = \frac{2n}{(n^2 + 1)}.$$

To abridge, the wave amplitude is

$$\rho_{\leftrightarrow} = \begin{cases} 
\rho_{1\leftrightarrow} = a_1 & \text{for } -\infty < z < 0, \\
\rho_{2\leftrightarrow} = \frac{a_1}{n} \sqrt{1 + (n^2 - 1) \cos^2 (k_2 z)} & \text{for } 0 < z < z_{23}, \\
\rho_{3\leftrightarrow} = \frac{a_1}{n} \sqrt{1 + (n^4 - 1) \cos^2 (k_1 z + \frac{\pi}{2} \left( \frac{n-1}{n} \right))} & \text{for } z_{23} < z < \infty.
\end{cases}$$

These functions are depicted in figure 2. The initial condition requesting that there is no reflected wave in medium 1 is tantamount to imposing the condition that the incident wave comes from medium 3.
5. Periodic $\lambda/4$ medium

5.1. Propagation in medium 3

Consider that the medium remains with refractive index $n_1$ until the plane $z_{31}$ is reached so that the optical path is again $\lambda/4$. The amplitude is then

$$\rho_{3\leftrightarrow} (z_{31}) = \frac{1}{\sqrt{\mu_3}} \sqrt{a_3^2 - b_3^2} \sqrt{1 + (\mu_3^2 - 1) \cos^2 (k_3 z_{31} + \varphi_{23})}$$

Let the phase accumulated in layer 3 to be $\frac{\pi}{2}$, then $k_1 z_{31} - k_1 z_{23} = k_0 n_1 (z_{31} - z_{23}) = \frac{\pi}{2}$, but $\varphi_{23} = \frac{\pi}{2} - k_1 z_{23}$ and the sum of the two layers is $k_1 z_{31} + \varphi_{23} = \pi$. Since $\sqrt{a_3^2 - b_3^2} = a_1$, the amplitude is then

$$\rho_{3\leftrightarrow} (z_{31}) = \frac{a_1}{\sqrt{\mu_3}} \sqrt{1 + (\mu_3^2 - 1) \cos^2 (\pi)} = a_1 n.$$  \hspace{1cm} (25)

Thereafter, the refractive index pattern is repeated. The nonlinear periodicity Floquet theorem for the nonlinear amplitude differential equation is [16]

$$\rho (z + d) = \frac{\rho (z)}{\sqrt{\rho_d}} \sqrt{1 + (\rho_d^2 - 1) \cos^2 \left( \frac{Q}{\rho^2 (z)} dz + \varphi_d \right)},$$  \hspace{1cm} (26)

where $d$ is the period. The periodicity in this case is

$$d = z_2 + z_3 = \frac{1}{k_0} \frac{\pi}{2} \left( \frac{1}{n_2} + \frac{1}{n_1} \right).$$

Since $\rho_{3\leftrightarrow} (z_{31}) = \rho_{1\leftrightarrow} (d)$, the amplitude (25) and the nonlinear periodicity Floquet equation (26) with $z = 0$ should be equal. Recall that $\rho (0) = a_1$ and the initial phase is zero. The periodic coefficient is then

$$\rho_d = \mu_3 = n_2^2.$$
whereas the periodic argument is
\[ \varphi_d = \pi. \]

Therefore, the amplitude between contiguous periods is
\[
\rho(z + d) = \frac{\rho(z)}{n} \sqrt{1 + (n^4 - 1) \cos^2 \left( \int \frac{Q}{\rho^2(z)} \, dz + \pi \right)}, \tag{27}
\]
whereas the amplitude for segments \( N \) periods apart is
\[
\rho(z + Nd) = \frac{\rho(z)}{n^N} \sqrt{1 + (n^4N - 1) \cos^2 \left( \int \frac{Q}{\rho^2(z)} \, dz + N\pi \right)}. \tag{28}
\]

5.2. **Period two, \( \lambda/4 \) layers**

For region 2 at \( (z + d) \), the amplitude is given by \( \rho_{2\leftrightarrow}(z) \) in (24) for \( 0 < z < z_{23} \), and from the periodic solutions relationship (27),
\[
\rho_{2\leftrightarrow}(z + d) = \frac{a_1}{n} \sqrt{1 + (n^2 - 1) \cos^2 (k_2 z)} \left( \frac{1}{n} \sqrt{1 + (n^4 - 1) \cos^2 (k_2 z + \pi)} \right).
\]

For region 3 at \( (z + d) \), the amplitude is \( \rho_{3\leftrightarrow}(z) \) in (24) for \( z_{23} < z < d \), and from (27),
\[
\rho_{3\leftrightarrow}(z + d) = \frac{a_1}{n} \sqrt{1 + (n^4 - 1) \cos^2 \left( \frac{k_1 z + \pi}{n} \left( \frac{n - 1}{n} \right) \right)} \left( \frac{1}{n} \sqrt{1 + (n^4 - 1) \cos^2 \left( \frac{k_1 z + \pi}{n} \left( \frac{n - 1}{n} \right) + \pi \right)} \right).
\]

These functions are plotted in figure 3. The amplitude modulation increases with respect to the single layer depicted in figure 2, thus revealing an increase in the reflectivity. The transmitted amplitude for unit incident amplitude
\[
a_1 = \left( 1 - r_0^2 \right)^{1/2} = \left( 1 - \left( \frac{n^4}{n^4 + 1} \right)^2 \right)^{1/2} = \frac{2n^2}{n^4 + 1},
\]
decreases accordingly.

5.3. **Period \( N+1, \lambda/4 \) layers**

For region 2 at \( (z + Nd) \), the amplitude is always \( \rho_{2\leftrightarrow}(z) \) given by (24) for \( 0 < z < z_{23} \). However, the periodic solutions factor varies according to (28),
\[
\rho_{2\leftrightarrow}(z + d) = \frac{a_1}{n} \sqrt{1 + (n^2 - 1) \cos^2 (k_2 z)} \left( \frac{1}{n^N} \sqrt{1 + (n^4N - 1) \cos^2 (k_2 z + \pi)} \right). \tag{29}
\]
Figure 3. Amplitude vs. distance as wave propagates through two λ/4 layers with n_2 = 1.8 (in green) separated by another λ/4 layer with n_1 = 1.3. The refractive indices are n = n_2/n_1 ≈ 1.385. The incident wave comes from the right with unit amplitude.

Similarly, for region 3 at (z + Nd), the amplitude is always ρ_{3→} (z) given by (24) for z_23 < z < d, but again, the periodic solutions factor varies according to (28),

$$\rho_{3→} (z + d) = \frac{a_1}{n} \sqrt{1 + (n^4 - 1) \cos^2 \left( k_1 z + \frac{\pi}{2} \left( \frac{n - 1}{n} \right) \right)}$$

$$\left( \frac{1}{n^N} \sqrt{1 + (n^{4N} - 1) \cos^2 \left( k_1 z + \frac{\pi}{2} \left( \frac{n - 1}{n} \right) + \pi \right)} \right)$$  (30)

Since N = 1 involves two periods, the reflectivity is

$$r_{3(N+1)} = \frac{b_{3(N+1)}}{a_{3(N+1)}} = \frac{\rho_{3→}^{(N+1)} - 1}{\rho_{3→}^{(N+1)} + 1} = \frac{n^{2(N+1)} - 1}{n^{2(N+1)} + 1},$$  (31)

whereas the normalization condition for N + 1 pairs of λ/4 layers (or periods) is

$$a_1 = \left( 1 - r_{2(N+1)}^2 \right)^{\frac{1}{2}} \left( 1 - \left( \frac{n^{2(N+1)} - 1}{n^{2(N+1)} + 1} \right)^2 \right)^{\frac{1}{2}} = \frac{2n^{(N+1)}}{n^{2(N+1)} + 1}.$$  (32)

6. Conclusions

The amplitude and phase formalism has been shown to reproduce in a simple and hopefully elegant analytical fashion, reflection, refraction and multiple beam interference. The philosophy of the method is as follows: The wave equation is translated into amplitude and phase differential equations. In doing so, unphysical potentials are avoided [17]. The general solution to the amplitude equation in homogeneous regions is proposed, that is, counter-propagating waves. The energy exchange invariant is invoked as a constant throughout the trajectory. Continuity of piecewise solutions are imposed to fulfill the differential equations. In this way, reflection, refraction and interference are incorporated, thus making unnecessary, for example, the sum of multiple reflections.

A dielectric mirror has been studied starting with a single interface and thereafter with a quarter-wavelength layer. The reflectivity as a function of the wavelength has not been included here but
can certainly be described with the present method. The multi-layered periodic structure has been tackled using a nonlinear version of Floquet’s theorem. The method, as we have illustrated in figure 4 and other figures in section 5, permits the analytic evaluation of the amplitude (or the actual field if desired) as it propagates in the multi-layered structure. All the variables involved have a straightforward physical meaning, they are either amplitudes, phases or quotients of the sum and difference of these quantities. Important design quantities such as the reflectivity are immediately obtained without further ado.

References

REFERENCES
