Orthogonal functions invariant for the time-dependent harmonic oscillator

M. Fernández Guasti a, *, A. Gil-Villegas b

a Departamento de Física, CBI, Universidad Autónoma Metropolitana-Iztapalapa, Michoacán y Purísima sn, Col Vicentina, 09340 México D.F., AP 55-534, Mexico
b Instituto de Física de la Universidad de Guanajuato, Lomas del Bosque 103, Fraccionamiento Lomas del Campestre, CP 37150, León, Guanajuato, Mexico

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Abstract

The Lewis invariant for the time-dependent harmonic oscillator is derived using a polar complex representation of the solution. This derivation is shown to be equivalent to an invariant stemming from the linearly independent solutions. The physical meaning of the involved constants and the associated auxiliary equation are elucidated.

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Time-dependent harmonic oscillators are described by the differential equation
\[ \ddot{q} + \Omega^2(t)q = 0, \]
where \( q \) is a canonical coordinate and \( \Omega(t) \) is an arbitrary continuous real function of time. The most general invariant of such a linear system, whose Hamiltonian is a homogeneous quadratic form in \( q \) and \( \dot{q} \), is
\[ I = \frac{1}{2} \left[ \frac{q^2}{\rho^2} + (\rho \dot{q} - q \dot{\rho})^2 \right], \]
where \( \rho \) is an auxiliary function of time that satisfies the Ermakov equation \( \ddot{\rho} + \Omega^2(t)\rho = 1/\rho^3 \). This invariant has been derived using Kruskal theory, symplectic group transformations and Noether’s theorem. The former procedure proves that a previously known adiabatic invariant is an exact invariant [1]. The second method invokes a time-dependent canonical transformation, which makes the Hamiltonian constant [2]. In the latter approach, Noether’s theorem relates the conserved quantities of a Lagrangian system to the symmetry groups, which leave the action invariant [3].

Let us derive this invariant using a remarkably elementary procedure. Allow for a polar representation of the canonical coordinate \( q(t) = Ae^{is} \), where the amplitude \( A \) and phase \( s \) are real time-dependent functions. Substitution in the time-dependent harmonic oscillator equation (1) leads to two equations for the real and imaginary parts:
\[ \frac{d^2A}{dt^2} - A \left( \frac{ds}{dt} \right)^2 = -\Omega^2A \quad \text{and} \]
\[ \frac{d^2A}{dt^2} + A \left( \frac{ds}{dt} \right)^2 = -\Omega^2A \]

* Corresponding author.
E-mail address: mfg@xanum.uam.mx (M. Fernández Guasti).
2i \frac{dA}{dt} \frac{ds}{dt} + iA \frac{d^2s}{dt^2} = 0; \quad (3)

the latter equation, provided that \( A \) is not zero, may be rewritten as

\[ \frac{1}{A} \frac{d}{dt} \left( A^2 \frac{ds}{dt} \right) = 0. \]

Thus, there exists an exact invariant given by

\[ Q = A^2 \frac{ds}{dt}. \quad (4) \]

An equation for the amplitude may be obtained from substitution in (3):

\[ \frac{d^2A}{dt^2} - \frac{Q^2}{A^3} = -\Omega^2 A; \quad (5) \]

this is the Ermakov equation stated before. The constant \( Q \) may be set equal to one when the amplitude is renormalized to a dimensionless amplitude. The auxiliary equation then represents the amplitude of the motion when the trajectory is described by the wave-like polar representation. The most general solution to the linear nonautonomous equation (1) is the linear superposition of two terms with opposite phases. This general solution, up to a multiplicative constant, may be written as

\[ q_g = q_+ + \sigma q_- = A_+ e^{is_+} + \sigma A_+ e^{-is_+}, \quad (6) \]

where \( \sigma \) is a constant that represents the quotient of the amplitudes with opposite phase. The above result may be rewritten as a single exponential by turning to the complex numbers additive representation, evaluating the sum and then returning to the polar representation:

\[ q_g(t) = Ae^{is} = A_+ \sqrt{1 + \sigma^2 + 2\sigma \cos(2s_+)} \times \exp \left[ i \arctan \left( \frac{1 - \sigma}{1 + \sigma} \tan s_+ \right) \right]; \quad (7) \]

thus, the invariant from Eq. (4) is

\[ Q = A_+^2 \left(1 + \sigma^2 + 2\sigma \cos(2s_+)\right) \times \frac{d}{dt} \arctan \left( \frac{1 - \sigma}{1 + \sigma} \tan s_+ \right) \]

\[ = A_+^2 \left(1 - \sigma^2\right) \frac{ds_+}{dt}. \quad (8) \]

This invariant is then proportional to the difference between the opposite phase wave intensities. Let us rewrite the invariant \( Q \) in terms of the amplitude \( A_+ \) and the canonical coordinate \( q_g \) in order to compare it with previous results. From Eq. (6),

\[ (A_+ \dot{q}_g - q_g \dot{A}_+)^2 = -A_+^4 s_+^2 \left(e^{2is_+} + \sigma^2 e^{-2is_+} - 2\sigma\right) \]

and

\[ q_g^2 / A_+^2 = e^{2is_+} + \sigma^2 e^{-2is_+} + 2\sigma, \]

the invariant \( Q \) may thus be written as

\[ Q^2 = (A_+ \dot{q}_g - q_g \dot{A}_+)^2 \frac{(1 - \sigma^2)^2}{4\sigma - q_g^2 / A_+^2}. \quad (9) \]

This representation is somewhat awkward because it mixes the complex variable general solution \( q_g \) and the amplitude of the positive phase solution \( A_+ \). The constants \( Q \) and \( \sigma \) may be related by making \( Q = 1 - \sigma^2 \). If we then solve for \( 2\sigma \), we obtain

\[ I = 2\sigma = \frac{1}{2} \left( \frac{q_g^2}{A_+^2} + (A_+ \dot{q}_g - q_g \dot{A}_+)^2 \right); \quad (10) \]

this quantity corresponds to the invariant derived by other methods in Eq. (2). According with the linear superposition (6), it is clear that \( I \) represents twice the amplitude ratio of the opposite phase solutions. If this invariant is zero it means that the boundary conditions are such that the opposite phase solution is zero. This interpretation is perhaps better visualized when it is examined in the spatial realm. In electromagnetic theory, one-dimensional inhomogeneous transparent media obey a similar harmonic oscillator formalism for monochromatic waves in the spatial domain [4]. Counter propagating infinite wave trains are the spatial counterpart of the above description where \( \sigma \) is the ratio of the counter propagating amplitudes.

Let us appraise several consequences of the present mathematical derivation. Recall the starting point of the derivation where the canonical coordinate was expressed as a complex number. This quantity may be equally expressed in the complex additive notation \( q = q_1 + q_2i \). Consequently, due to the linearity of the time-dependent harmonic oscillator equation, two real functions \( q_1 \) and \( q_2 \) must obey this equation. Therefore, it is being requested that two variables, and not just one, fulfill the harmonic equation. The invariant \( Q \) may be derived within this formalism using the following real algebra procedure. The product of the differential equation of one function, say \( \dot{q}_2 + \Omega^2(t)q_2 = 0 \) times the other function \( q_1 \), yields
\[ q_1 \ddot{q}_2 + \Omega^2(t) q_1 q_2 = 0. \] Evaluation of the variables the other way around gives \[ q_2 \ddot{q}_1 + \Omega^2(t) q_2 q_1 = 0. \] From their difference, we obtain

\[ q_1 \ddot{q}_2 - q_2 \ddot{q}_1 = 0. \]

This equation may be written in terms of a first-order derivative as

\[ \frac{d}{dt}(q_1 \dot{q}_2 - q_2 \dot{q}_1) = 0 \]

and thus the invariant

\[ Q_{12} = q_1 \dot{q}_2 - q_2 \dot{q}_1 \]

is obtained. This quantity is zero if the functions are a linear combination of each other. Since they both satisfy the same differential equation, the only non-vanishing possibility is that there is a phase shift between them. Allow then for two functions with an arbitrary phase shift between them

\[ q_1 = A \cos s \quad \text{and} \quad q_2 = \xi_0 A \cos s \pm \xi_{90} A \sin s. \]

The constants \( \xi_0 \) and \( \xi_{90} \) are the in and out of phase contributions of \( q_2 \) with respect to \( q_1 \). The invariant is then

\[ Q_{12} = A \cos s \left( -\xi_0 A \dot{s} \sin s + \xi_{90} \dot{A} \cos s \right) \]

\[ + (\xi_0 A \cos s \pm \xi_{90} A \sin s) \times (-A \dot{s} \sin s + \dot{A} \cos s) \]

\[ = \pm \xi_{90} \dot{A}^2 \dot{s}. \]

which is equivalent to the invariant \( Q \) derived in Eq. (4). The only nonvanishing contribution to the invariant \( Q_{12} \) comes from functions that are 90° out of phase or in other terms, from the linearly independent solutions of the harmonic equation. It is in this sense that we refer to orthogonal functions. This result is reminiscent of the Sturm–Liouville theory and the orthogonality of functions. In that case, the weighted integral of two functions is the continuum analog of the outer product of the vector terms where a null result exhibits that the functions are orthogonal. In the present case, the differential form \( q_1 \ddot{q}_2 - q_2 \ddot{q}_1 \) is equivalent to the continuum analog of the inner product where a null result exhibits that the functions are linear combinations of each other or parallel in the vector terminology.

Recall that the energy of a system executing small oscillations with constant frequency \( \omega \) is given by

\[ W = \frac{1}{2} m A^2 \omega^2. \]

If the time-dependent parameter \( \Omega^2 \) varies slowly compared with the period of oscillation, the time derivative of the phase may be approximated by the frequency,

\[ \dot{s} = \frac{d}{dt}[\omega(t)] \approx \omega(t) \]

and the time-dependent energy written as

\[ W(t) \approx \frac{1}{2} m A^2(t) \omega^2(t). \]

The exact invariant \( Q \) may then be estimated by

\[ Q = A^2 \dot{s} \approx A^2 \omega \approx \frac{2 W(t)}{m \omega(t)}. \]

This approximate expression is proportional to the adiabatic invariant of a mechanical system with slowly varying parameters [5]. In contrast with the invariant derived by Lewis and subsequent workers, the orthogonal functions invariant \( Q \) is proportional, in the adiabatic approximation, to the energy per unit frequency of the oscillator.

References